module-01

Laplace transforms

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Definition:

If $f(t)$ is a real valued function defined for all $t \geq 0$ then the Laplace transform of $f(t)$ denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_{t=0}^{\infty} e^{-st} f(t)\, dt$$

provided the integral exists. on integration of the indefinite integral we will be having a function of s and t. when this is evaluated between the limits $t=0$ and $t=\infty$ we will be left with a function of s only and we shall denote it by $\bar{f}(s)$, where s is a parameter, real or complex. thus $L[f(t)] = \bar{f}(s)$

Equivalently we can express this in the form

$$L^{-1}[\bar{f}(s)] = f(t)$$

and is called the inverse laplace transform.

NOTE: $L[c_1 f_1(t) \pm c_2 f_2(t)] = c_1 L[f_1(t)] \pm c_2 L[f_2(t)]$

where c_1 and c_2 are constants

Bernoulli's Rule

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots\cdots$$

where u', u'', u''', u^{iv} are successive differentiation $v_1, v_2, v_3, v_4 \cdots$ are successive integrals of v.

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where $V_1 = \int V\,dx$, $V_2 = \int V_1\,dx$

Example

① Find $L[f(t)]$ where $f(t) = \begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$

Sol^n: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L[f(t)] = \int_0^4 e^{-st} f(t)\,dt + \int_4^\infty e^{-st} f(t)\,dt$$

Using the relevant $f(t)$ in the integrals we have

$$L[f(t)] = \int_0^4 e^{-st}\, t\,dt + \int_4^\infty e^{-st}\cdot 5\,dt$$

$$= \int_0^4 t\cdot e^{-st}\,dt + 5\int_4^\infty e^{-st}\,dt$$

Using Bernoulli's rule for the first term in RHS we have,

$$L[f(t)] = \left[t\cdot\frac{e^{-st}}{-s} - (1)\left(\frac{1}{-s}\right)\frac{e^{-st}}{-s}\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= -\frac{1}{s}\left[te^{-st} + \frac{e^{-st}}{s}\right]_0^4 - \frac{5}{s}\left[e^{-st}\right]_4^\infty$$

$$= -\frac{1}{s}\left[\left\{4e^{-4s} + \frac{e^{-4s}}{s}\right\} - \left\{0 + \frac{e^0}{s}\right\}\right] - \frac{5}{s}\left[e^{-\infty} - e^{-4s}\right]$$

$$= -\frac{1}{s}\left[4e^{-4s} + \frac{e^{-4s}}{s} - \frac{1}{s}\right] - \frac{5}{s}\left[0 - e^{-4s}\right]$$

$$= -\frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{1}{s^2} + \frac{5e^{-4s}}{s}$$

$$L[f(t)] = \frac{e^{-4s}}{s} + \frac{1}{s^2}\left(1 - e^{-4s}\right)$$

② Find $L[f(t)]$ if $f(t) = \begin{cases} \sin 2t, & 0 < t < \pi \\ 0, & t > \pi \end{cases}$

Solⁿ: $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[f(t)] = \int_0^\pi e^{-st} f(t)\, dt + \int_\pi^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \sin 2t\, dt + \int_\pi^\infty e^{-st}\, 0 . dt$$

W.K.T $\int e^{at} \sin bt\, dt = \frac{e^{at}}{a^2+b^2}(a \sin bt - b \cos bt)$

$$= \left[\frac{e^{-st}}{(-s)^2+2^2}(-s \sin 2t - 2\cos 2t)\right]_{t=0}^{\pi} + 0$$

$$= \frac{-1}{s^2+4}\left[e^{-st}(+s\sin 2t + 2\cos 2t)\right]_{t=0}^{\pi}$$

$$= \frac{-1}{s^2+4}\left[e^{-s\pi}(s\sin 2\pi + 2\cos 2\pi) - e^{0}(s\sin 2(0) + 2\cos 2(0))\right]$$

$\sin 2\pi = 0 = \sin 0$

$\cos 2\pi = 1 = \cos 0$

$$= \frac{-1}{s^2+4}\left[e^{-s\pi}(0 + 2(1)) - (1)(0 + 2(1))\right]$$

$$= \frac{-1}{s^2+4}\left[e^{-s\pi}(2) - 2\right]$$

$$= \frac{-2}{s^2+4}\left[e^{-s\pi} - 1\right] = \frac{2}{s^2+4}\left[1 - e^{-s\pi}\right]$$

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$\therefore L[f(t)] = \frac{2}{s^2+4}\left[1-e^{-s\pi}\right]$

Laplace transform of elementary functions

① $L(a)$ where a is constant

$L[f(t)] = L(a) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$

$= \int_0^\infty e^{-st} \cdot a \, dt$

$= a\left[\frac{e^{-st}}{-s}\right]_0^\infty$

$= \frac{a}{-s}\left[e^{-st}\right]_0^\infty$

$= a/_{-s}\left[e^{-\infty} - e^{0}\right]$

$= a/_{-s}\left[0-1\right]$

$L(a) = a/s$, where $s>0$

If $a=1$,
eg. $L(1) = 1/s$

If $a=2$, $L[2] = 2/s$

If $a=100$, $L[100] = \frac{100}{s}$

② $L(e^{at})$

$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$

$L[e^{at}] = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$

$= \int_0^\infty e^{-(s-a)t} \cdot dt$

$$L[e^{at}] = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[0-1\right]$$

$$L[e^{at}] = \frac{1}{s-a}$$

NOTE: $L[e^{-at}] = \frac{1}{s+a}$, $L[e^{-2t}] = \frac{1}{s+2}$

$L[e^{2t}] = \frac{1}{s-2}$

$L[e^{3t}] = \frac{1}{s-3}$

$$L[\cosh at] = \int_0^\infty e^{-st} \frac{e^{at}+e^{-at}}{2}$$

③ $L(\cosh at)$

$$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right)$$

$$\frac{1}{2}\int_0^\infty e^{-st}\cdot e^{at} + e^{-st}$$

$$= \frac{1}{2}\left\{L(e^{at}) + L(e^{-at})\right\}$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t} + e^{-(s+a)} dt$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)}}{s-a} + \frac{e^{-(s+a)}}{s+a}\right]_0^\infty$$

$$= \frac{1}{2}\left\{\frac{s+\cancel{a}+s-\cancel{a}}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{2}\left[0+0+\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$= \frac{1}{\cancel{2}}\left\{\frac{\cancel{2}s}{s^2-a^2}\right\}$$

$$= \frac{1}{2}\left(\frac{2s}{s^2-a^2}\right)$$

$$= \frac{s}{s^2-a^2}$$

Thus $L(\cosh at) = \frac{s}{s^2-a^2}$, where $s>a$

④ $L(\sinh at)$

$$L(\sinh at) = L\left(\frac{e^{at} - e^{-at}}{2}\right)$$

$$= \frac{1}{2}\left\{L(e^{at}) - L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\}$$

$$= \frac{1}{2}\left\{\frac{(s+a) - (s-a)}{(s-a)(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2}\right\}$$

$$= \frac{1}{\cancel{2}}\left\{\frac{\cancel{2}a}{s^2-a^2}\right\}$$

$$= \frac{a}{s^2-a^2}$$

$$L(\sinh at) = \frac{a}{s^2-a^2}, \text{ where } s>a$$

NOTE: $L(\sinh 2t) = \frac{2}{s^2-4}$

$$L(\sinh 3t) = \frac{3}{s^2-9}$$

$$L(\sinh t) = \frac{1}{s^2-1}$$

⑤ $L(\cos at)$

$$L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cos at \cdot dt$$

using $\int e^{at}\cos bt\,dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$

$$L(\cos at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[e^{-st}(-s\cos at + a\sin at)\right]_{t=0}^{\infty}$$

$$= \frac{1}{s^2+a^2}\left[0 - e^{0}(-s\cos(0) + a\sin(0)\right]$$

$$= \frac{1}{s^2+a^2}\left[+s(1) - 0\right]$$

$$L[\cos at] = \frac{s}{s^2+a^2} \quad \text{where } s>0$$

⑥ $L(\sin at)$

$$L(\sin at) = \int_0^{\infty} e^{-st}\cdot \sin at\,dt$$

using, $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}(a\sin bt - b\cos bt)$

$$L(\sin at) = \left[\frac{e^{-st}}{(-s)^2+a^2}(-s\sin at - a\cos at)\right]_{t=0}^{\infty}$$

$$L(\sin at) = \frac{-1}{s^2+a^2}\left[e^{-st}(s\sin at + a\cos at)\right]_{t=0}^{\infty}$$

$$= \frac{-1}{s^2+a^2}\left[0 - e^{0}(s\sin(0) + a\cos(0)\right]$$

$$= \frac{1}{s^2+a^2}\left[(1)(0+a)\right] \Rightarrow L(\sin at) = \frac{a}{s^2+a^2}$$

(7) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$, where n is constant

w.k.t $\Gamma(n+1) = n!$, if n is a +ve integer

$L(t^n) = \frac{n!}{s^{n+1}}$, If n is a +ve integer.

Table of Laplace transforms

	$f(t)$	$L[f(t)] = \bar{f}(s)$		$f(t)$	$L[f(t)] = \bar{f}(s)$
1.	a	a/s	5	$\sinh at$	$\frac{a}{s^2-a^2}$
2.	e^{at}	$\frac{1}{s-a}$	6	$\sin at$	$\frac{a}{s^2+a^2}$
3.	$\cosh at$	$\frac{s}{s^2-a^2}$	7	t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$
4.	$\cos at$	$\frac{s}{s^2+a^2}$	8	t^n $n=1,2,3\ldots$	$\frac{n!}{s^{n+1}}$

problems

(1) Find the Laplace transform of the following functions

(i) $\cosh^2 3t$

Soln: Let, $f(t) = \cosh^2 3t = (\cosh 3t)^2$

$$f(t) = \left[\frac{e^{3t} + e^{-3t}}{2}\right]^2$$

$$\cancel{f(t) = 1/4 [(e^{3t})^2 + (e^{-3t})^2]}$$

$$\cancel{= 1/4 [e^{6t} + e^{6t}}$$

$$f(t) = \frac{1}{2^2}\left[e^{3t} + e^{-3t}\right]^2$$

$\hookrightarrow (a+b)^2$

$$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{3t} e^{-3t}\right]$$

$$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{3t-3t}\right]$$

$$= \frac{1}{4}\left[e^{6t} + e^{-6t} + 2e^{0}\right]$$

$$f(t) = \frac{1}{4}\left[e^{6t} + e^{-6t} + 2\right]$$

$$L[f(t)] = \frac{1}{4} L\left[e^{6t} + e^{-6t} + 2\right]$$

$$= \frac{1}{4}\left\{L(e^{6t}) + L(e^{-6t}) + L(2)\right\}$$

$$L[f(t)] = \frac{1}{4}\left\{\frac{1}{s-6} + \frac{1}{s+6} + \frac{2}{s}\right\}$$

Do yourself

* $\sinh^2 2t$

(2) $e^{-2t}\sinh 4t$

Soln: Let, $f(t) = e^{-2t}\sinh 4t$

w.k.t $\sinh t = \frac{e^{t} - e^{-t}}{2}$

$$\sinh 4t = \frac{e^{4t} - e^{-4t}}{2}$$

$$f(t) = e^{-2t}\sinh 4t = e^{-2t}\left[\frac{e^{4t} - e^{-4t}}{2}\right]$$

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$$= \frac{1}{2}\left[e^{-2t}\cdot e^{4t} - e^{-2t}\cdot e^{-4t}\right]$$

$$= \frac{1}{2}\left[e^{2t} - e^{-6t}\right]$$

$$f(t) = \frac{1}{2}\left[e^{2t} - e^{-6t}\right]$$

$$L[f(t)] = \frac{1}{2}\left[L(e^{2t}) - L(e^{-6t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{S-2} - \frac{1}{S+6}\right]$$

$$= \frac{1}{2}\left[\frac{(S+6)-(S-2)}{(S-2)(S+6)}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{S}+6-\cancel{S}+2}{(S-2)(S+6)}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{8}\,4}{(S-2)(S+6)}\right]$$

$$L[f(t)] = \frac{4}{(S-2)(S+6)}$$

③ $\sin 5t \cdot \cos 2t$

Solⁿ: Let $f(t) = \sin 5t \cos 2t$

W.K.T $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$f(t) = \frac{1}{2}\left[\sin(5t+2t) + \sin(5t-2t)\right]$$

$$f(t) = \frac{1}{2}\left[\sin 7t + \sin 3t\right]$$

$$L[f(t)] = \frac{1}{2}\left[L(\sin 7t) + L(\sin 3t)\right]$$

w.k.T $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{7}{s^2+7^2} + \frac{3}{s^2+3^2}\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

(4) $\cos t \cdot \cos 2t \cdot \cos 3t$

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Sol^{n}: Let $f(t) = \cos t \cos 2t \cos 3t$

w.k.T $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$\cos t \cos 2t = \frac{1}{2}[\cos(t+2t) + \cos(t-2t)]$$

$$= \frac{1}{2}[\cos 3t + \cos(t)]$$

$$= \frac{1}{2}[\cos 3t + \cos t]$$

$$\therefore \cos t \cos 2t \cos 3t = \frac{1}{2}[\cos 3t + \cos t]\cos 3t$$

$$= \frac{1}{2}[\cos 3t \cos 3t + \cos t \cos 3t]$$

$$= \frac{1}{2}\left[\frac{1}{2}[\cos(3t+3t) + \cos(3t-3t)] + \frac{1}{2}[\cos(t+3t) + \cos(t-3t)]\right]$$

$$= \frac{1}{2} \times \frac{1}{2}\left[(\cos 6t + \cos 0) + (\cos 4t + \cos(-2t))\right]$$

$$f(t) = \frac{1}{4}\left[\cos 6t + 1 + \cos 4t + \cos 2t\right]$$

$$\mathcal{L}[f(t)] = \frac{1}{4}\left[\mathcal{L}[\cos 6t] + \mathcal{L}[1] + \mathcal{L}[\cos 4t] + \mathcal{L}[\cos 2t]\right]$$

w.k.T $\mathcal{L}[\cos at] = \frac{s}{s^2+a^2}$, $\mathcal{L}[a] = a/s$

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$$= \frac{1}{4}\left[\frac{s}{s^2+6^2} + \frac{1}{s} + \frac{s}{s^2+4^2} + \frac{s}{s^2+2^2}\right]$$

$$L[f(t)] = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{1}{s} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

(5) $\sin^2(2t+1)$

Sol^{n}: Let $f(t) = \sin^2(2t+1)$

w.k.t $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$f(t) = \sin^2(2t+1)$$

$$= \frac{1-\cos 2(2t+1)}{2}$$

$$= \frac{1}{2}\left[1-\cos(4t+2)\right]$$

$$= \frac{1}{2}\left[1-\{\cos 4t \cos 2 - \sin 4t \sin 2\}\right]$$

$$f(t) = \frac{1}{2}\left[1 - \cos 4t \cos 2 + \sin 4t \sin 2\right]$$

$$L[f(t)] = \frac{1}{2}\left[L(1) - L(\cos 4t)\cos 2 + \sin 2\, L(\sin 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\cos 2 + \sin 2 \cdot \frac{4}{s^2+4^2}\right]$$

$$L[f(t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s\cdot\cos 2}{s^2+16} + \frac{4\sin 2}{s^2+16}\right]$$

⑥ $(3t+4)^3 + 5^t$

Sol: Let $f(t) = (3t+4)^3 + 5^t$

w.k.T $(a+b)^3 = a^3 + b^3 + 3a^2b + 3ab^2$

$$(3t+4)^3 = (3t)^3 + 4^3 + 3(3t)^2(4) + 3(3t)(4)^2 + e^{\log 5 \cdot t}$$

$$= 27t^3 + 64 + 108t^2 + 144t + e^{\log 5 \cdot t}$$

$$L[(3t+4)^3] = 27L[t^3] + L[64] + 108L[t^2] + 144L[t] + L[e^{\log 5 \cdot t}]$$

$$= 27 \cdot \frac{3!}{s^4} + \frac{64}{s} + 108 \cdot \frac{2!}{s^3} + 144 \cdot \frac{1}{s^2} + \frac{1}{s - \log 5}$$

$$= \frac{162}{s^4} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{64}{s} + \frac{1}{s - \log 5}$$

⑦ $f(t) = 3\sqrt{t} + \frac{4}{\sqrt{t}}$

$$f(t) = 3t^{1/2} + 4/t^{1/2}$$

$$f(t) = 3t^{1/2} + 4t^{-1/2}$$

w.k.T $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

$$L[f(t)] = L[3t^{1/2}] + L[4t^{-1/2}]$$

$$= 3L[t^{1/2}] + 4L[t^{-1/2}]$$

$$= 3\left[\frac{\Gamma(1/2+1)}{s^{1/2+1}}\right] + 4\left[\frac{\Gamma(-1/2+1)}{s^{-1/2+1}}\right]$$

$$L[f(t)] = 3\frac{\Gamma(3/2)}{s^{3/2}} + 4\frac{\Gamma(1/2)}{s^{1/2}}$$

$\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$

$$L[f(t)] = 3\frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} + 4\frac{\sqrt{\pi}}{s^{1/2}}$$

$$= 3\frac{\sqrt{\pi}}{2s^{3/2}} + \frac{4\sqrt{\pi}}{s^{1/2}}$$

$$= \frac{3}{2}\frac{\sqrt{\pi}}{s^{1/2}\cdot s} + \frac{4\sqrt{\pi}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[\frac{3}{2s} + 4\right]$$

$$L[f(t)] = \sqrt{\frac{\pi}{s}}\left[\frac{3}{2s} + 4\right]$$

Properties of Laplace transforms

① If $L[f(t)] = \bar{f}(s)$ then $L[e^{at} f(t)] = \bar{f}(s-a)$

② If $L[f(t)] = \bar{f}(s)$, then

$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$ where n is a +ve integer

In particular $L[tf(t)] = -\frac{d}{ds}[\bar{f}(s)]$

$L[t^2 f(t)] = \frac{d^2}{ds^2}[\bar{f}(s)]$

③ If $L[f(t)] = \bar{f}(s)$, then $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

Problems

* Find the Laplace transform of the following functions

① $e^{-2t}(2\cos 5t - \sin 5t)$

Soln: Let $f(t) = 2\cos 5t - \sin 5t$

$$L[f(t)] = 2L[\cos 5t] - L[\sin 5t]$$

$$= 2\cdot\frac{s}{s^2+5^2} - \frac{5}{s^2+5^2}$$

$$L[f(t)] = \frac{2s-5}{s^2+5^2}$$

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$$\mathcal{L}\left[e^{-2t} f(t)\right] = \left\{\frac{2s-5}{s^2+25}\right\}_{s \to s+2}$$

$$= \frac{2(s+2)-5}{(s+2)^2+25}$$

$$= \frac{2s+4-5}{s^2+2^2+4s+25}$$

$$\mathcal{L}\left[e^{-2t} f(t)\right] = \frac{2s-1}{s^2+4s+29}$$

$$\therefore \mathcal{L}\left[e^{-2t}(2\cos 5t - \sin 5t)\right] = \frac{2s-1}{s^2+4s+29}$$

(2) $e^{-t} \cos^2 3t$

Sol^n: W.K.T $\cos^2 t = \frac{1+\cos 2t}{2}$

$$f(t) = \cos^2 3t = \frac{1+\cos 6t}{2}$$

$$\mathcal{L}\left[f(t)\right] = \mathcal{L}\left[\frac{1+\cos 6t}{2}\right]$$

$$= \frac{1}{2}\mathcal{L}\left[1+\cos 6t\right]$$

$$= \frac{1}{2}\left\{\mathcal{L}[1] + \mathcal{L}[\cos 6t]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2+6^2}\right\}$$

$$\mathcal{L}\left[f(t)\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

$$\mathcal{L}\left[e^{-t}\cos^2 3t\right] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]_{s \to s+1}$$

$$\mathcal{L}\left[e^{-t}\cos^2 3t\right] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+36}\right]$$

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③ $(1+3te^{2t})^2$

Sol^n: Let $f(t) = (1+3te^{2t})^2$

use $(a+b)^2 = a^2+b^2+2ab$

$(1+3te^{2t})^2 = 1+(3te^{2t})^2 + 2(1)(3te^{2t})$

$(1+3te^{2t})^2 = 1+9t^2e^{4t} + 6te^{2t}$

$\mathcal{L}[f(t)] = \mathcal{L}[1+9t^2e^{4t} + 6te^{2t}]$

$= \mathcal{L}[1] + 9\mathcal{L}[e^{4t} \cdot t^2] + 6\mathcal{L}[te^{2t}]$

$= \frac{1}{s} + 9\mathcal{L}[t^2]_{s\to s-4} + 6\mathcal{L}[t]_{s\to s-2}$

But $\mathcal{L}(t) = 1/s^2$ $\mathcal{L}(t^2) = 2/s^3$

use $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\mathcal{L}[(1+3te^{2t})^2] = 1/s + 9 \cdot \frac{2}{(s-4)^3} + \frac{6}{(s-2)^2}$

$\mathcal{L}[(1+3te^{2t})^2] = 1/s + \frac{18}{(s-4)^3} + \frac{6}{(s-2)^2}$

④ $\sinh at \sin at$

Sol^n: $f(t) = \sinh at \sin at$

$f(t) = \frac{e^{at}-e^{-at}}{2} \cdot \sin at$

$f(t) = 1/2 (e^{at}\sin at - e^{-at}\sin at)$

$\mathcal{L}[f(t)] = 1/2 \left\{ \mathcal{L}[\sin at]_{s\to s-a} - \mathcal{L}[\sin at]_{s\to s+a} \right\}$

But $\mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$

$= 1/2 \left\{ \frac{a}{(s-a)^2+a^2} - \frac{a}{(s+a)^2+a^2} \right\}$

$= 1/2 \left\{ \frac{a}{s^2+a^2-2as+a^2} - \frac{a}{s^2+a^2+2as+a^2} \right\}$

$$= \frac{a}{2}\left\{\frac{1}{s^2+2a^2-2as} + \frac{1}{s^2+2a^2+2as}\right\}$$

$$= \frac{a}{2}\left\{\frac{(s^2+2a^2+2as)-(s^2+2a^2-2as)}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right\}$$

$$= \frac{a}{2}\left\{\frac{2as+2as}{s^4+4a^4}\right\}$$

$$= \frac{a}{2}\left\{\frac{4as}{s^4+4a^4}\right\}$$

$(s^2+2a^2-2as)(s^2+2a^2+2as)$

$s^4 + \cancel{2s^2a^2} + \cancel{2as^3} +$

$\cancel{2s^2a^2} + 4a^4 + \cancel{4a^3s}$

$\cancel{-2as^3} - \cancel{4a^3s} -$

$\cancel{4a^2s^2}$

$s^4 + 4a^4$

(8) $\sin^3 2t \cosh t$

Sol^n: Let $f(t) = \sin^3 2t$

$= \frac{1}{4}\{3\sin 2t + \sin$

W.K.T $\sin 3t = 3\sin t - 4\sin^3 t$

$4\sin^3 t = 3\sin t - \sin 3t$

$\sin^3 t = \frac{1}{4}[3\sin t - \sin 3t]$

$\sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 3(2t)]$

$\sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 6t]$

$$L[\sin^3 2t] = \frac{1}{4}\{L[3\sin 2t] - L[\sin 6t]\}$$

$$= \frac{1}{4}\{3L[\sin 2t] - L[\sin 6t]\}$$

$$= \frac{1}{4}\left\{3\cdot\frac{2}{s^2+4} - \frac{6}{s^2+36}\right\}$$

$$= \frac{1}{4}\left\{\frac{6}{s^2+4} - \frac{6}{s^2+36}\right\}$$

$$= \frac{1}{4}\left\{\frac{6(s^2+36)-6(s^2+4)}{(s^2+4)(s^2+36)}\right\}$$

$$= \frac{1}{4}\left\{\frac{6s^2+216-6s^2-24}{(s^2+4)(s^2+36)}\right\}$$

$$= \frac{1}{4}\left\{\frac{192}{(s^2+4)(s^2+36)}\right\}$$

$$= \frac{48}{(s^2+4)(s^2+36)}$$

Now, $\mathcal{L}(\cosh t \sin^3 2t) = \mathcal{L}\left[\frac{e^t+e^{-t}}{2} \cdot \sin^3 2t\right]$

$$= \frac{1}{2}\left\{\mathcal{L}[e^t \cdot \sin^3 2t] + \mathcal{L}[e^{-t} \cdot \sin^3 2t]\right\}$$

$$= \frac{1}{2}\left\{\mathcal{L}[\sin^3 2t]_{s\to s-1} + \mathcal{L}[\sin^3 2t]_{s\to s+1}\right\}$$

$$= \frac{1}{2}\left[\frac{48}{((s-1)^2+4)((s-1)^2+36)} + \frac{48}{((s+1)^2+4)((s+1)^2+36)}\right]$$

$$= \frac{48}{2}\left[\frac{1}{((s-1)^2+4)((s-1)^2+36)} + \frac{1}{((s+1)^2+4)((s+1)^2+36)}\right]$$

$$= 24\left[\frac{1}{(s^2-2s+5)(s^2-2s+37)} + \frac{1}{(s^2+2s+5)(s^2+2s+37)}\right]$$

Find the Laplace transform of the following functions

① $t\cos at$

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Soln: Let $f(t) = \cos at$

$$\mathcal{L}[f(t)] = \frac{s}{s^2+a^2}$$

Now $\mathcal{L}[t f(t)] = (-1)^1\frac{d}{ds}[\bar{f}(s)]$

$$\mathcal{L}[t\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left\{\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right\}$$

$$= -\left\{\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right\}$$

$$= -\left\{\frac{a^2-s^2}{(s^2+a^2)^2}\right\}$$

$$\mathcal{L}[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$ //

② $t^2 \sin at$

Sol^n: let $f(t) = \sin at$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin at]$$

$$= \frac{a}{s^2+a^2}$$

Now $\mathcal{L}[t^2 f(t)] = (-1)^2\frac{d^2}{ds^2}[\bar{f}(s)]$

$$= \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{d}{ds}\left\{\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right\}$$

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$$= \frac{d}{ds}\left\{\frac{(s^2+a^2)(0) - a(2s)}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{-2as}{(s^2+a^2)^2}\right\}$$

$$= \left\{\frac{(s^2+a^2)^2(-2a) - (-)2as\,[2(s^2+a^2)(2s)}{((s^2+a^2)^2)^2}\right\}$$

$$= \frac{\cancel{(s^2+a^2)}\,2a\left\{-(s^2+a^2) + 4s^2\right\}}{\cancel{(s^2+a^2)^4}}$$

$$L[t^2 \sin at] = \frac{2a\left\{-(s^2+a^2) + 4s^2\right\}}{(s^2+a^2)^3}$$

$$= \frac{2a\left\{-s^2 - a^2 + 4s^2\right\}}{(s^2+a^2)^3}$$

$$= \frac{2a\left\{3s^2 - a^2\right\}}{(s^2+a^2)^3}$$

③ $t^3 \sin t$

Sol^n: Let $f(t) = \sin t$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] \qquad \because \mathcal{L}[\sin at] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[f(t)] = \frac{1}{s^2+1}$$

$\longrightarrow$ property ②

Now, $$\mathcal{L}[t^3 f(t)] = (-1)^3 \frac{d^3}{ds^3}[\bar{f}(s)]$$

$$= (-1)^3 \frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{(s^2+1)(0)-(1)(2s)}{(s^2+1)^2}\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= +\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{2s}{(s^2+1)^2}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)^2(2) - 2s\left(2(s^2+1)(2s)\right)}{\left((s^2+1)^2\right)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^4+1+2s^2)(2) - 8s^2(s^2+1)}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{2s^4+2+4s^2-8s^4-8s^2}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{-6s^4-4s^2+2}{(s^2+1)^4}\right]$$

$$= \frac{d}{ds}\left[\frac{\cancel{(s^2+1)}\left\{2(s^2+1) - 8s^2\right\}}{\cancel{(s^2+1)^4}}\right]$$

$$= \frac{d}{ds}\left[\frac{2s^2+2-8s^2}{(s^2+1)^3}\right]$$

$$= \frac{d}{ds}\left[\frac{2-6s^2}{(s^2+1)^3}\right]$$

$$= 2\frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left\{\frac{(s^2+1)^3(0-6s) - (1-3s^2)(3(s^2+1)^2(2s))}{((s^2+1)^3)^2}\right\}$$

$$= 2\left\{\frac{(s^2+1)^3(-6s) - 6s(1-3s^2)(s^2+1)^2}{(s^2+1)^6}\right\}$$

$$= 2\left\{\frac{\cancel{(s^2+1)^2}\left\{-6s(s^2+1) - 6s(1-3s^2)\right\}}{\cancel{(s^2+1)^6}}\right\}$$

$$= 2\left\{\frac{-6s\left\{s^2+1+1-3s^2\right\}}{(s^2+1)^4}\right\}$$

$$= \frac{-12s(2-2s^2)}{(s^2+1)^4}$$

$$= \frac{-24s(1-s^2)}{(s^2+1)^4} \Rightarrow \mathcal{L}\left[t^3 f(t)\right] = \frac{24s(s^2-1)}{(s^2+1)^4}$$

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④ $t^3 \cosh t$

Solⁿ: Let $f(t) = t^3 \cosh t$

NOTE: Just for understanding: Here we should not prefer to work the problem IIIly to previous, since we have $\cosh t$ which can be converted to $\frac{e^t + e^{-t}}{2}$ so that it will be highly convenient to apply the shifting property.

$$f(t) = t^3 \left(\frac{e^t + e^{-t}}{2} \right)$$

$$f(t) = \frac{1}{2} \left\{ t^3 e^t + t^3 e^{-t} \right\}$$

$$f(t) = \frac{1}{2} \left\{ e^t \cdot t^3 + e^{-t} \cdot t^3 \right\} \rightarrow \text{use property ①}$$

$$\mathcal{L}[f(t)] = \frac{1}{2} \left\{ \mathcal{L}[t^3]_{s \to s-1} + \mathcal{L}[t^3]_{s \to s+1} \right\}$$

w.k.t $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$ $\therefore \mathcal{L}[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$$\mathcal{L}[f(t)] = \frac{1}{2} \left\{ \frac{6}{(s-1)^4} + \frac{6}{(s+1)^4} \right\}$$

$$= \frac{6}{2} \left\{ \frac{1}{(s-1)^4} + \frac{1}{(s+1)^4} \right\}$$

$$\mathcal{L}[f(t)] = 3 \left\{ \frac{1}{(s-1)^4} + \frac{1}{(s+1)^4} \right\}$$

⑤ $t^5 e^{4t} \cosh 3t$

Solⁿ: $f(t) = t^5 e^{4t} \cosh 3t$

$$f(t) = t^5 e^{4t} \left\{ \frac{e^{3t} + e^{-3t}}{2} \right\}$$

$$f(t) = \frac{e^{7t} \cdot t^5 + e^{t} \cdot t^5}{2}$$

$$f(t) = \frac{1}{2} \left\{ e^{7t} \cdot t^5 + e^{t} \cdot t^5 \right\}$$

$$\mathcal{L}[f(t)] = \frac{1}{2} \left\{ \mathcal{L}[t^5]_{s \to s-7} + \mathcal{L}[t^5]_{s \to s-1} \right\} \text{ (by property ①)}$$

W.K.T $L[t^n] = \frac{n!}{s^{n+1}}$

$$L[t^5] = \frac{5!}{s^{5+1}}$$

$$L[t^5] = \frac{120}{s^6}$$

$$L[t^5 e^{4t}\cosh 3t] = \frac{1}{2}\left\{\frac{120}{(s-7)^6} + \frac{120}{(s-1)^6}\right\}$$

$$= \frac{120}{2}\left\{\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right\}$$

$$L[t^5 e^{4t}\cosh 3t] = 60\left\{\frac{1}{(s-7)^6} + \frac{1}{(s-1)^6}\right\}$$

⑥ $t e^{-2t} \sin 4t$

Soln: let $f(t) = t e^{-2t} \sin 4t$

$$L[\sin 4t] = \frac{4}{s^2+16}$$

$$\therefore L[e^{-2t} \sin 4t] = L\left[\frac{4}{s^2+16}\right]_{s \to s+2}$$

$$= \frac{4}{(s+2)^2+16}$$

$$L[e^{-2t}\sin 4t] = \frac{4}{s^2+4+4s+16}$$

$$L[e^{-2t}\sin 4t] = \frac{4}{s^2+4s+20}$$

$$L[t e^{-2t}\sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2+4s+20}\right]$$

$$= -\left\{\frac{(s^2+4s+20)(0) - 4(2s+4)}{(s^2+4s+20)^2}\right\}$$

$$= \frac{4(2s+4)}{(s^2+4s+20)^2}$$

$$= \frac{8(s+2)}{(s^2+4s+20)^2}$$

⑦ S.T $\int_0^\infty t^3 e^{-t} \sin t \, dt = 0$

Solⁿ: W.K.T $\int_0^\infty e^{-st} f(t)\, dt = \mathcal{L}[f(t)]$

$$\int_0^\infty e^{-st} t^3 \sin t \, dt = \mathcal{L}[t^3 \sin t] \quad \text{—— ①}$$

Now we have to find $\mathcal{L}[t^3 \sin t]$, $f(t) = \sin t$

$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$$\mathcal{L}[t^3 \sin t] = -\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right] \quad \{\text{property ②}\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}\right\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{(s^2+1)(0)-(1)(2s)}{(s^2+1)^2}\right\}$$

$$= -\frac{d^2}{ds^2}\left\{\frac{-2s}{(s^2+1)^2}\right\} = \frac{d^2}{ds^2}\left\{\frac{2s}{(s^2+1)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{d}{ds}\left\{\frac{2s}{(s^2+1)^2}\right\}\right\}$$

$$= \frac{d}{ds}\left\{\frac{(s^2+1)^2(2) - 2s(2(s^2+1)(2s))}{(s^2+1)^4}\right\}$$

$$= \frac{d}{ds}\left\{\frac{\cancel{(s^2+1)}(2)\{(s^2+1)-4s^2\}}{\cancel{(s^2+1)^4}}\right\}$$

$$= 2\frac{d}{ds}\left\{\frac{s^2+1-4s^2}{(s^2+1)^3}\right\}$$

$$= 2\frac{d}{ds}\left\{\frac{1-3s^2}{(s^2+1)^3}\right\}$$

$$= 2\left\{\frac{(s^2+1)^3(-6s)-(1-3s^2)\left(3(s^2+1)^2(2s)\right)}{\left((s^2+1)^3\right)^2}\right\}$$

$$= 2\left\{\cancel{(s^2+1)^2}(-6s)\left\{\frac{(s^2+1)+(1-3s^2)}{\cancel{(s^2+1)^6}}\right\}\right\}$$

$$= -12s\left\{\frac{-2s^2+2}{(s^2+1)^4}\right\}$$

$$= -24s\left\{\frac{-s^2+1}{(s^2+1)^4}\right\}$$

$$= 24s\left\{\frac{s^2-1}{(s^2+1)^4}\right\}$$

Thus ① becomes

$$\int_0^\infty e^{-st}\,t^3\sin t\,dt = 24s\left\{\frac{s^2-1}{(s^2+1)^4}\right\}$$

put $s=1$

$$\int_0^\infty e^{-t}\,t^3\sin t\,dt = 24\left\{\frac{1-1}{(2)^4}\right\}$$

$$= 24(0)$$

$$\int_0^\infty e^{-t}\,t^3\sin t\,dt = 0$$

⑧ S.T. $\int_0^{\infty} t e^{-2t} \sin 4t \, dt = 1/25$

Solⁿ: W.K.T $\int_0^{\infty} e^{-st} f(t)\, dt$ and $\mathcal{L}[f(t)]$

$$\int_0^{\infty} e^{-st}\, t \sin 4t \, dt = \mathcal{L}[t \sin 4t] \rightarrow ①$$

$\mathcal{L}[t\sin 4t] \rightarrow$ {property ②}

$f(t) = \sin 4t,\ \mathcal{L}[f(t)] = \mathcal{L}[\sin 4t] = \frac{4}{s^2+16}$

$$\mathcal{L}[t\sin 4t] = -\frac{d}{ds}\left[\frac{4}{s^2+16}\right]$$

$$= -\left[\frac{(s^2+16)(0) - 4(2s)}{(s^2+16)^2}\right]$$

$$\mathcal{L}[t\sin 4t] = \frac{8s}{(s^2+16)^2}$$

$$① \Rightarrow \int_0^{\infty} e^{-st}\, t \sin 4t = \frac{8s}{(s^2+16)^2}$$

Thus, put $s = 2$

$$\int_0^{\infty} e^{-2t}\, t \sin 4t = \frac{8 \times 2}{(2^2+16)^2}$$

$$\int_0^{\infty} e^{-2t}\, t \sin 4t = 16/400$$

$$\int_0^{\infty} e^{-2t}\, t \sin 4t = 1/25$$

Doyourself

* Find the value of $\int_0^\infty t e^{-3t} \cos 2t \, dt$ using Laplace transform.

Ans: $\int_0^\infty e^{-3t} \cdot t \cos 2t \, dt = \frac{5}{169}$

Find the Laplace transform of the following functions

① $\frac{1-e^{-at}}{t}$

Sol: Use property 3. $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s) \, ds$

Let $f(t) = 1 - e^{-at}$

$$L[f(t)] = L[1 - e^{-at}]$$

$$= L[1] - L[e^{-at}]$$

$$= \frac{1}{s} - \frac{1}{s+a}$$

$$\therefore L[f(t)] = \frac{1}{s} - \frac{1}{s+a}$$

$$L[f(t)] = \bar{f}(s)$$

we have $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s) \, ds$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s+a}\right) ds$$

$$= \left[\log s - \log(s+a)\right]_s^\infty$$

$\hookrightarrow \{\log m - \log n = \log(m/n) \text{ form}\}$

$$= \left[\log\left(\frac{s}{s+a}\right)\right]_s^{\infty}$$

$$= \left[\lim_{s\to\infty} \log\left(\frac{s}{s+a}\right)\right] - \log\left(\frac{s}{s+a}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{s(1+a/s)}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \log\left(\frac{1}{1+0}\right) - \log\left(\frac{s}{s+a}\right)$$

$$= \log(1) - \log\left(s/s+a\right)$$

$$= 0 - \log\left(s/s+a\right)$$

$$= \log\left(\frac{s+a}{s}\right)$$

$\log a^{-m} = \log\frac{1}{a^m}$

$$-\log\left(\frac{s}{s+a}\right) = \log\left(\frac{s}{s+a}\right)^{-1} = \log\left(\frac{s+a}{s}\right)$$

(2) $\frac{\sin^2 t}{t}$

Solⁿ: Let $f(t) = \sin^2 t$

$$f(t) = \frac{1-\cos 2t}{2}$$

$$\therefore L[f(t)] = L\left[\frac{1-\cos 2t}{2}\right]$$

$$= \frac{1}{2}\left\{L[1] - L[\cos 2t]\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$$

$$L[f(t)] = \bar{f}(s)$$

hence $L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\, ds$

$$= \frac{1}{2}\int_s^{\infty}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\} ds$$

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$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\left\{\log s - \frac{1}{2}\log(s^2+4)\right\}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$$

w.k.t $m\log n = \log n^m$

$$= \frac{1}{2}\left[\log s - \log(s^2+4)^{1/2}\right]_s^\infty$$

$\log m - \log n = \log(m/n)$

$$= \frac{1}{2}\left[\log\left(\frac{s}{(s^2+4)^{1/2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2(1+4/s^2)}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{\cancel{s}}{\cancel{s}\sqrt{1+4/s^2}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log\left[\frac{1}{\sqrt{1+0}}\right] - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= \frac{1}{2}\log(1) - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$= 0 - \frac{1}{2}\log\left[\frac{s}{\sqrt{s^2+4}}\right]$$

$$\mathcal{L}\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\log\left[\frac{\sqrt{s^2+4}}{s}\right]$$

③ $\frac{2\sin t \sin 5t}{t}$

Solⁿ: let $f(t) = 2\sin t \sin 5t$

w.k.T $\sin A \sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$

$\sin t \sin 5t = -\frac{1}{2}\left[\cos 6t - \cos(-4t)\right]$

$= -\frac{1}{2}\left[\cos 6t - \cos 4t\right]$

$\sin t \sin 5t = \frac{1}{2}\left[\cos 4t - \cos 6t\right]$

Now, $f(t) = \cancel{2} \cdot \frac{1}{\cancel{2}}\left[\cos 4t - \cos 6t\right]$

$f(t) = \cos 4t - \cos 6t$

$L[f(t)] = L[\cos 4t] - L[\cos 6t]$

$= \frac{s}{s^2+16} - \frac{s}{s^2+36}$

$L[f(t)] = \bar{f}(s)$

$L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

$= \int_s^\infty \left\{\frac{s}{s^2+16} - \frac{s}{s^2+36}\right\} ds$

$= \left[\frac{1}{2}\log(s^2+16) - \frac{1}{2}\log(s^2+36)\right]_s^\infty$

$= \frac{1}{2}\left[\log(s^2+16) - \log(s^2+36)\right]_s^\infty$

$= \frac{1}{2}\left[\log\left(\frac{s^2+16}{s^2+36}\right)\right]_s^\infty$

$= \lim_{s\to\infty} \frac{1}{2}\log\left[\frac{\cancel{s^2}(1+16/s^2)}{\cancel{s^2}(1+36/s^2)}\right] - \frac{1}{2}\log\left[\frac{s^2+16}{s^2+36}\right]$

$$= \frac{1}{2}\left\{\cancel{\log(1)}^{0} - \log\left(\frac{s^2+16}{s^2+36}\right)\right\}$$

$$= \frac{1}{2}\left\{-\log\left(\frac{s^2+16}{s^2+36}\right)\right\}$$

$$= \frac{1}{2}\log\left(\frac{s^2+36}{s^2+16}\right)$$

$$\Rightarrow m\log n = \log n^m$$

$$= \log\left(\frac{s^2+36}{s^2+16}\right)^{1/2}$$

$$\mathcal{L}\left[\frac{2\sin t\sin 5t}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}$$

④ $\frac{\sin at}{t}$

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Solⁿ: Let $f(t) = \sin at$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin at]$$

$$\mathcal{L}[f(t)] = \frac{a}{s^2+a^2}$$

$$\mathcal{L}[f(t)] = \bar{f}(s)$$

hence $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$\mathcal{L}\left[\frac{\sin at}{t}\right] = a \times \frac{1}{a}\left[\tan^{-1}(s/a)\right]_s^\infty$$

$$\frac{d}{dx}(\tan^{-1}x/a) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1}x/a) = \frac{1}{1+\frac{x^2}{a^2}} \times \frac{1}{a}$$

$$= \frac{a}{a^2+x^2}$$

$= \frac{\alpha}{a}\left[\tan^{-1}(s/a)\right]_s^{\infty}$

$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$

$= \pi/2 - \tan^{-1}(s/a)$

W.K.T

$\cot^{-1}x = \pi/2 - \tan^{-1}x$

$\tan\pi/2 = \infty$

$\therefore \pi/2 = \tan^{-1}\infty$

$L\left[\frac{\sin at}{t}\right] = \cot^{-1}(s/a)$

⑥ ACU Jan 2020 $\frac{\cos at - \cos bt}{t}$

Solⁿ: Let $f(t) = \cos at - \cos bt$

$L[f(t)] = L[\cos at - \cos bt]$

$= L[\cos at] - L[\cos bt]$

$L[f(t)] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$L[f(t)] = \bar{f}(s)$

$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

$= \int_s^{\infty}\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$

$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^{\infty}$

$= \frac{1}{2}\left[\log\left\{\frac{s^2+a^2}{s^2+b^2}\right\}\right]_s^{\infty}$

$= \frac{1}{2}\left\{\lim_{s\to\infty}\log\left(\frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

$= \frac{1}{2}\left\{\cancel{\log(1)}^{0} - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

$\frac{1}{a}$ $= \frac{1}{2}\left\{-\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

Scanned with CamScanner

$$= \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$= \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

$$\mathcal{L}\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$$

NOTE: * $m\log n = \log n^m$

eg: $-\log(s/s+a) = \log(s/s+a)^{-1}$

$$= \log\left(\frac{1}{\frac{s}{s+a}}\right)$$

$$-\log(s/s+a) = \log\left(\frac{s+a}{s}\right)$$

* $\log(mn) = \log m + \log n$

* $\log(m/n) = \log m - \log n$

Doyourself

* Find the Laplace transform of $\frac{\sinh t}{t}$

Ans: $\mathcal{L}\left[\frac{\sinh t}{t}\right] = \log\sqrt{\frac{s+1}{s-1}}$

* Find the Laplace transform of $2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$

June 2017 Soln: The given function be denoted by $f(t)$ and Let,

$$f(t) = F(t) + G(t) + H(t)$$

where $F(t) = 2^t$,

$$G(t) = \frac{\cos 2t - \cos 3t}{t}$$

$$H(t) = t\sin t$$

$$\therefore \mathcal{L}[f(t)] = \mathcal{L}[F(t)] + \mathcal{L}[G(t)] + \mathcal{L}[H(t)] \quad \text{—— ①}$$

Now, $\mathcal{L}[F(t)] = \mathcal{L}[2^t]$

$$= \mathcal{L}[e^{\log 2 \cdot t}]$$

$$\mathcal{L}[F(t)] = \frac{1}{s-\log 2} \qquad \left[\because e^{\log 2} = 2\right]$$

So $e^{\log 2 \cdot t} = 2^t$

$$G(t) = \frac{\cos 2t - \cos 3t}{t}$$

we property ③ $\longrightarrow \mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

$$\mathcal{L}[G(t)] = \int_s^\infty \bar{f}(s)\,ds$$

$$= \int_s^\infty \mathcal{L}[\cos 2t - \cos 3t]\,dt$$

$$= \int_s^\infty \left[\frac{s}{s^2+4} - \frac{s}{s^2+9}\right] dt$$

$$= \left[\tfrac{1}{2}\log(s^2+4) - \tfrac{1}{2}\log(s^2+9)\right]_s^\infty$$

$$= \tfrac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]_s^\infty$$

$$= \left\{\tfrac{1}{2}\log\left[\frac{s^2+4}{s^2+9}\right]\right\}_s^\infty$$

tsint

$$= \left[\log\left[\frac{s^2+4}{s^2+9}\right]^{1/2}\right]_s^\infty$$

$$= \left[\log\sqrt{\frac{s^2+4}{s^2+9}}\right]_s^\infty$$

$$\cancel{= \left[\log\sqrt{\frac{s^2(1+4/s^2)}{s^2(1+9/s^2)}}\right]_s^\infty}$$

$= \cancel{\log}$

$$= \lim_{s\to\infty} \log\sqrt{\frac{s^2+4}{s^2+9}} - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \lim_{s\to\infty} \log\sqrt{\frac{s^2(1+4/s^2)}{s^2(1+9/s^2)}} - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \log\sqrt{\frac{1+0}{1+0}} - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \log(1) - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= 0 - \log\sqrt{\frac{s^2+4}{s^2+9}}$$

$$= \log\left(\sqrt{\frac{s^2+4}{s^2+9}}\right)^{-1}$$

$$= \log\left(\frac{1}{\sqrt{\frac{s^2+4}{s^2+9}}}\right)$$

$$\mathcal{L}[g(t)] = \log\left(\sqrt{\frac{s^2+9}{s^2+4}}\right)$$

$H(t) = t\sin t$ $\longrightarrow$ $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\bar{f}(s)]$

use property ②

$$\mathcal{L}[H(t)] = \mathcal{L}[t\sin t] = (-1)\frac{d}{ds}(\bar{f}(s))$$

but

$$= -\frac{d}{ds}[\mathcal{L}(\sin t)]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{\cancel{(s^2+1)(0)} - (1)(2s)}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

eq^n ① becomes

$$\mathcal{L}[f(t)] = \frac{1}{s-\log 2} + \log\sqrt{s^2+9/s^2+4} + \frac{2s}{(s^2+1)^2}$$

* Find the Laplace transform of $t^2 e^{-3t} \sin 2t$

Sol^n: we shall find first $\mathcal{L}(t^2 \sin 2t)$

$$\mathcal{L}[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}(\sin 2t)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\left[\frac{2}{s^2+4}\right]\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$$

$$= \left[\frac{(s^2+4)^2(-4) - (-4s)(2)(s^2+4)(2s)}{((s^2+4)^2)^2}\right]$$

$$= \cancel{(s^2+4)}\left\{\frac{-4(s^2+4) + 16s^2}{\cancel{(s^2+4)^4}}\right\}$$

$$= \frac{16s^2 - 4s^2 - 16}{(s^2+4)^3}$$

$$\mathcal{L}[t^2 \sin 2t] = \frac{12s^2-16}{(s^2+4)^3}$$

$$\mathcal{L}[e^{-3t} t^2 \sin 2t] = \left[\frac{12s^2-16}{(s^2+4)^3}\right]_{s \to s+3} \quad \{\text{property ①}\}$$

$$\mathcal{L}[e^{-3t} t^2 \sin 2t] = \frac{12(s+3)^2 - 16}{((s+3)^2+4)^3}$$

Laplace transform of periodic functions

~~Statements~~

Definition: A function $f(t)$ is said to be a periodic function of period $T>0$ if $f(t+nT)=f(t)$ where $n=1,2,3 \ldots$

ex: $\sin t$, $\cos t$ are periodic functions of period 2π because
$\sin(t+2n\pi)=\sin t$, $\cos(t+2n\pi)=\cos t$

Theorem: If $f(t)$ is a periodic function of period T, then $\boxed{L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt}$

[without proof]

Problems

① If $f(t)=t^2$, $0<t<2$ and $f(t+2)=f(t)$ for $t>2$, find $L[f(t)]$.

Soln: $f(t)$ is a periodic function of period 2.

$\therefore T=2$

we have $L[f(t)]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$

$$L[f(t)]=\frac{1}{1-e^{-s2}}\int_0^2 e^{-st}\,t^2\,dt$$

$$L[f(t)]=\frac{1}{1-e^{-2s}}\int_0^2 t^2.e^{-st}\,dt$$

Apply Bernoulli's rule of integration by parts

② Dec 2017 18

So

$$\cancel{\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\left\{\left[t^2\frac{e^{-st}}{-s} - 2t\times\frac{1}{-s}\times\frac{e^{-st}}{-s} + 2\cdot\frac{1}{s^2}\times\frac{e^{-st}}{-s}\right]_0^2\right.}$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2s}}\left[t^2\frac{e^{-st}}{-s} - 2t\cdot\frac{1}{-s}\times\frac{e^{-st}}{-s} + 2\cdot\frac{1}{s^2}\times\frac{e^{-st}}{-s}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left[t^2\cdot\frac{e^{-st}}{-s} - \frac{2t\,e^{-st}}{s^2} - \frac{2}{s^3}e^{-st}\right]_0^2$$

$$= \frac{1}{1-e^{-2s}}\left[\left\{2^2\cdot\frac{e^{-2s}}{-s} + \frac{2(2)e^{-2s}}{s^2} - \frac{2}{s^3}e^{-2s}\right\} - \left\{0-0-\frac{2}{s^3}\right\}\right]$$

$\boxed{dt}$

$$= \frac{1}{1-e^{-2s}}\left[\frac{-4e^{-2s}}{+s} + \frac{4e^{-2s}}{s^2} - \frac{2}{s^3}e^{-2s} + \frac{2}{s^3}\right] \rightarrow \text{Lcm is } s^3$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[2e^{-2s}(-s^2) + 2e^{-2s}\cdot s - e^{-2s} + 1\right]$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[e^{-2s}(-2s^2 + 2s - 1) + 1\right]$$

$$= \frac{2}{s^3(1-e^{-2s})}\left[1 - e^{-2s}(2s^2 + 2s + 1)\right]$$

(2) Dec 2017, 18 — Find the Laplace transform of the full wave rectifier $f(t) = E\sin\omega t,\ 0 < t < \pi/\omega$ having period π/ω

Sol^n: we have $\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$f(t)$ is a periodic function of period π/ω

Now

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-s(\pi/\omega)}}\int_0^{\pi/\omega} e^{-st} E \sin\omega t \, dt.$$

$$\mathcal{L}[f(t)] = \frac{E}{1-e^{-\pi s/\omega}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$\because$ w.k.t $\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$\mathcal{L}[f(t)] = \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[\left\{e^{-s\pi/\omega}(-s\sin\omega\pi/\omega - \omega\cos\omega\pi/\omega) - e^0(0-\omega(1))\right\}\right]$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[\left\{e^{-s\pi/\omega}(-s\sin\pi - \omega\cos\pi) - (1)(-\omega)\right\}\right]$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[e^{-s\pi/\omega}(0-\omega(-1)) + \omega\right]$$

$$= \frac{E}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[e^{-s\pi/\omega}(\omega) + \omega\right]$$

$$= \frac{E\omega}{(1-e^{-\pi s/\omega})(s^2+\omega^2)}\left[1+e^{-s\pi/\omega}\right]$$

$\times^{ly}$ both Nr & Dr by $e^{\pi s/2\omega}$ in RHS

$$= \frac{E\omega(1+e^{-s\pi/\omega})(e^{\pi s/2\omega})}{(1-e^{-\pi s/\omega})(s^2+\omega^2)e^{\pi s/2\omega}}$$

$$= \frac{E\omega}{s^2+\omega^2} \frac{e^{\pi s/2\omega} + e^{-s\pi/\omega} \cdot e^{\pi s/2\omega}}{e^{\pi s/2\omega} - e^{-\pi s/\omega} \cdot e^{\pi s/2\omega}}$$

$$= e^{-s\pi/\omega + \pi s/2\omega}$$

$$= e^{-\frac{2s\pi + \pi s}{2\omega}}$$

$$= e^{-s\pi/2\omega}$$

$$= \frac{E\omega}{s^2+\omega^2} \cdot \frac{e^{\pi s/2\omega} + e^{-s\pi/2\omega}}{e^{\pi s/2\omega} - e^{-s\pi/2\omega}}$$

$$\left[\omega.K.T \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2} \right]$$

So X^{ly} and $\div$ both Nr & Dr by 2

$$\mathcal{L}[f(t)] = \frac{E\omega}{s^2+\omega^2} \times \frac{2\left(\frac{e^{\pi s/2\omega} + e^{-s\pi/2\omega}}{2}\right)}{2\left(\frac{e^{\pi s/2\omega} - e^{-s\pi/2\omega}}{2}\right)}$$

$$= \frac{E\omega}{s^2+\omega^2} \times \frac{\cancel{2}\cosh(\pi s/2\omega)}{\cancel{2}\sinh(\pi s/2\omega)}$$

$$\mathcal{L}[f(t)] = \frac{E\omega}{s^2+\omega^2} \coth(\pi s/2\omega)$$

(3) Given $f(t) = \begin{cases} E, & 0 < t < a/2 \\ -E & a/2 < t < a \end{cases}$ where $f(t+a) = f(t)$

Jan 2020 ACU

S.T $\mathcal{L}[f(t)] = E/s \cdot \tanh(as/4)$.

Solⁿ: The given function is periodic with period $T = a$

we have $\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$$= \frac{1}{1-e^{-sa}} \int_0^a e^{-st} f(t)\, dt$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st}\, E\, dt + \int_{a/2}^{a} e^{-st}(-E)\, dt\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\int_0^{a/2} e^{-st}\, dt - \int_{a/2}^{a} e^{-st}\, dt\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left[\left\{\frac{e^{-s\cdot a/2}}{-s} - \frac{e^{0}}{-s}\right\} + \left\{\frac{e^{-sa}}{s} - \frac{e^{-s\cdot a/2}}{s}\right\}\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\frac{e^{-s\cdot a/2}}{-s} + \frac{1}{s} + \frac{e^{-sa}}{s} - \frac{e^{-sa/2}}{s}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-e^{-sa/2} + 1 + e^{-sa} - e^{-sa/2}\right\}$$

$$= \frac{E}{s(1-e^{-sa})}\left\{-2e^{-sa/2} + 1 + e^{-sa}\right\}$$

$$= \frac{E}{s(1-e^{-sa})}\left\{1 + e^{-sa} - 2e^{-sa/2}\right\}$$

$\hookrightarrow a = 1,\ b = e^{-as/2}$ ~~~~ write it in the form of $(a-b)^2$

$$= \frac{E}{s(1-e^{-sa})}\left\{(1 - e^{-as/2})^2\right\}$$

$$\mathcal{L}[f(t)] = \frac{E\cancel{(1-e^{-as/2})}^{2}}{s\cancel{(1-e^{-as/2})}(1+e^{-as/2})}$$

$$\mathcal{L}[f(t)] = \frac{E(1-e^{-as/2})}{s(1+e^{-as/2})}$$

$\times^{ly}$ both Nr & Dr by $e^{as/4}$ we get

$$\mathcal{L}[f(t)] = \frac{E(1-e^{-as/2})e^{as/4}}{s(1+e^{-as/2})e^{as/4}}$$

$$= \frac{E(e^{as/4} - e^{-as/2}\cdot e^{as/4})}{s(e^{as/4}+e^{-as/2}\, e^{as/4})}$$

$$= \frac{E(e^{as/4} - e^{-as/2 + as/4})}{s(e^{as/4} + e^{-as/2 + as/4})}$$

$$= e^{-as/2 + as/4}$$

$$= e^{\frac{-as\cdot 2+as}{4}}$$

$$= e^{-as/4}$$

$$= \frac{E(e^{as/4} - e^{-as/4})}{s(e^{as/4}+e^{-as/4})}$$

$\times^{ly}$ & $\div$ by 2 both Nr & Dr

$$= \frac{E\,\cancel{2}\,\frac{(e^{as/4}-e^{-as/4})}{2}}{s\cdot\cancel{2}\,\frac{(e^{as/4}+e^{-as/4})}{2}}$$

w.k.T.

$$\sinh x = \frac{e^{x}-e^{-x}}{2}$$

$$\cosh x = \frac{e^{x}+e^{-x}}{2}$$

$$= \frac{E}{s} \times \frac{\sinh(as/4)}{\cosh(as/4)}$$

$$\mathcal{L}[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

Do yourself

(4) If $f(t) = \begin{cases} E, & 0<t<a \\ -E & a<t<2a \end{cases}$ S.T $\mathcal{L}[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$

H.W

June 2017

(5) If $f(t) = \begin{cases} t & 0 \le t \le a \\ 2a-t, & a \le t \le 2a \end{cases}$, $f(t+2a) = f(t)$

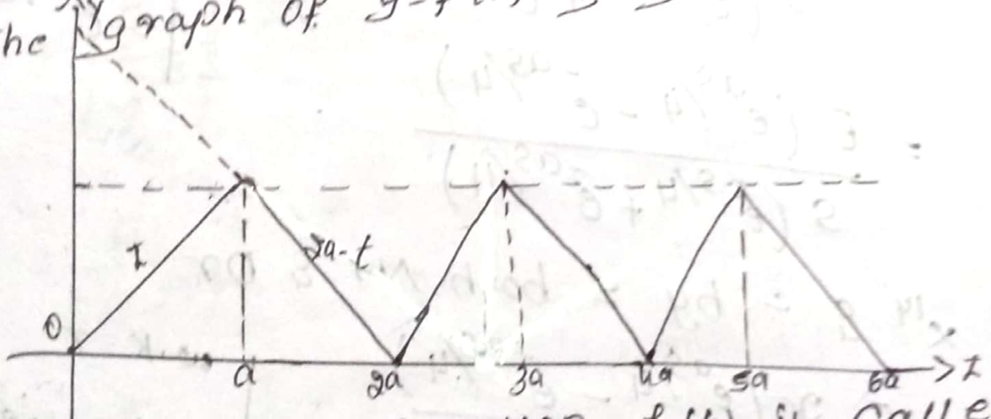
June & Dec 2016

(i) Sketch the graph of $f(t)$ as a periodic function

(ii) S.T $\mathcal{L}[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$

Solⁿ: (i) Let $f(t) = y$ and $y = t$ is a straight line passing through the origin making an angle 45° with the t-axis. $y = 2a - t$ or $y + t = 2a$ or $\frac{t}{2a} + \frac{y}{2a} = 1$ is a straight line passing through the points $(2a, 0)$ and $(0, 2a)$.

The graph of $y = f(t)$ is as follows



The periodic function $f(t)$ is called triangular wave function.

(ii) we have $T = 2a$ and $\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a t e^{-st}\,dt + \int_a^{2a} (2a-t) e^{-st}\,dt\right\}$$

Apply Bernoulli's rule

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\left[t\,\frac{e^{-st}}{-s} - (1)\frac{e^{-st}}{s^2}\right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_a^{2a}\right\}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left\{\left[\left(-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2}\right) - \left(0 - \frac{e^{0}}{s^2}\right)\right] + \left[\left((2a-2a)\frac{e^{-2as}}{s} - (-1)\frac{e^{-2as}}{s^2}\right) - \left((2a-a)\frac{e^{-2as}}{-s} + \frac{e^{-sa}}{s^2}\right)\right]\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{-\frac{ae^{-sa}}{s} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} + 0 + \frac{e^{-2as}}{s^2} - \left(-\frac{ae^{-as}}{s} + \frac{e^{-sa}}{s^2}\right)\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\cancel{-\frac{ae^{-sa}}{s}} - \frac{e^{-sa}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \cancel{\frac{ae^{-as}}{s}} - \frac{e^{-sa}}{s^2}\right\}$$

$$= \frac{1}{(1-e^{-2as})s^2}\left\{-e^{-sa} + 1 + e^{-2as} - e^{-sa}\right\}$$

$$= \frac{1}{s^2(1-e^{-2as})}\left\{1 - 2e^{-sa} + e^{-2as}\right\}$$

$\hookrightarrow a = 1$

$b = e^{-as}$

write in the form of $(a-b)^2$

$$= \frac{1}{s^2(1-e^{-2as})}\left\{(1-e^{-as})^2\right\}$$

$$\mathcal{L}[f(t)] = \frac{\cancel{(1-e^{-as})}^{2}}{s^2 \cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$\times^{ly}$ & $\div$ both Nr & Dr by $e^{as/2}$

$$= \frac{e^{as/2} - e^{-as}\, e^{as/2}}{s^2(e^{as/2} + e^{-as}\, e^{as/2})}$$

$$= \frac{e^{as/2} - e^{-as/2}}{s^2(e^{as/2} + e^{-as/2})}$$

$\times^{ly}$ & $\div$ both Nr & Dr by 2

$$= \frac{1}{s^2} \frac{\dfrac{\cancel{2}(e^{as/2} - e^{-as/2})}{2}}{\dfrac{\cancel{2}(e^{as/2} + e^{-as/2})}{2}}$$

$$= \frac{1}{s^2} \frac{\sinh(as/2)}{\cosh(as/2)}$$

$$= \frac{1}{s^2}\tanh(as/2)$$

$$\therefore \mathcal{L}[f(t)] = \frac{1}{s^2}\tanh(as/2)$$

Do yourself

June 2015
* Find $\mathcal{L}[f(t)]$ if $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2-t, & 1 \le t \le 2 \end{cases}$

June 2018
* Find $\mathcal{L}[f(t)]$ if $f(t) = \begin{cases} t, & 0 < t < \pi \\ \pi - t, & \pi < t < 2\pi \end{cases}$

⑥ A periodic function of period $2\pi/\omega$ is defined by $f(t) = \begin{cases} E\sin\omega t, & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega \le t \le 2\pi/\omega \end{cases}$ where E and ω are constant. S.T $\mathcal{L}[f(t)] = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

Solⁿ: we have for periodic function $f(t)$.

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad \text{Here } T = 2\pi/\omega$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\, dt$$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-2\pi s/\omega}} \left\{ \int_0^{\pi/\omega} e^{-st} E\sin\omega t\, dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} \cdot 0 \cdot dt \right\}$$

$$\mathcal{L}[f(t)] = \frac{E}{1-e^{-2\pi s/\omega}} \left\{ \int_0^{\pi/\omega} e^{-st} \sin\omega t\, dt \right\}$$

w.K.T $\displaystyle\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$$L[f(t)] = \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}\left(-s\sin\omega t - \omega\cos\omega t\right)\right]_0^{\pi/\omega}$$

$$= \frac{E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[\left\{e^{-s\pi/\omega}\left(-s\sin\omega\tfrac{\pi}{\omega} - \omega\cos\omega\tfrac{\pi}{\omega}\right)\right\} - \left\{e^{0}(0-\omega)\right\}\right]$$

$$= \frac{E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[-e^{-s\pi/\omega}(s\underbrace{\sin\pi}_{0} + \omega\cos\pi) + \omega\right]$$

$$= \frac{E}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[-e^{-s\pi/\omega}(\omega(-1)) + \omega\right]$$

$$= \frac{E\omega}{(s^2+\omega^2)(1-e^{-2\pi s/\omega})}\left[1+e^{-s\pi/\omega}\right]$$

$$(1-e^{-s\pi/\omega})(1+e^{-s\pi/\omega}) = 1+\cancel{e^{-s\pi/\omega}} - \cancel{e^{-s\pi/\omega}} - e^{-s\pi/\omega}\cdot e^{-s\pi/\omega} = 1-e^{-2s\pi/\omega}$$

$$= \frac{E\omega\cancel{(1+e^{-s\pi/\omega})}}{(s^2+\omega^2)(1-e^{-s\pi/\omega})\cancel{(1+e^{-s\pi/\omega})}}$$

$$L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1+e^{-s\pi/\omega})}$$

Unit Step function (Heaviside function)

Definition: The unit step function $u(t-a)$ or Heaviside function $H(t-a)$ is defined as follows

$$u(t-a) = \begin{cases} 0, & t \le a \\ 1, & t > a \end{cases}$$ where a is constant

Properties associated with the unit step function

① $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

② $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ where $\mathcal{L}[f(t)] = \bar{f}(s)$

NOTE: ① when $f(t-a) = 1$ as we have $f(t)$ also equal to 1 and hence $\mathcal{L}[f(t)] = 1/s$.

② $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

③ If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

④ If $f(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & a < t \le b \\ f_3(t), & t > b \end{cases}$

Then $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$.

Problems:-

① Working procedure

TYPE ①: To find $\mathcal{L}[F(t)u(t-a)]$ where $F(t)$ is a polynomial in t.

Step ①: Let $F(t) = f(t-a)$ which imply that $F(t+a) = f(t)$.

i.e Replace t by $t+a$ to obtain $f(t)$ and find $\mathcal{L}[f(t)] = \bar{f}(s)$

Step ②: $\mathcal{L}[F(t)\,u(t-a)] = \mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$

by property (ii)

Type ②: Given $f(t)$ as discontineous function, to find $\mathcal{L}[f(t)]$ by expressing $f(t)$ in terms of unit step function

Step ①: we express $f(t)$ in terms of unit step function by directly making use of result (iii) or (iv) as the case may be

Step ②: we find $\mathcal{L}[f(t)]$ as in Type 1

Problems

(*) Find the Laplace transform of the following functions

$\rightarrow \{F(t)\,u(t-a)\}$ form

① $[e^{t-1} + \sin(t-1)]\,u(t-1)$

Sol^n: Let $F(t) = e^{t-1} + \sin(t-1)$

$F(t) = f(t-1)$

$f(t-1) = [e^{t-1} + \sin(t-1)]$

To get $f(t)$, replace t by $t+1$

$F(t+1) = f(t)$

$f(t) = e^{t} + \sin t$

$\mathcal{L}[f(t)] = \mathcal{L}[e^{t} + \sin t]$

$= \mathcal{L}[e^{t}] + \mathcal{L}[\sin t]$

$= \frac{1}{s-1} + \frac{1}{s^2+1}$

$= \bar{f}(s)$

$$\mathcal{L}[f(t-1)u(t-1)] = e^{-s}\bar{f}(s) \quad (\because a=1)$$

$$\mathcal{L}[e^{t-1} + \sin(t-1)]u(t-1) = e^{-s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$$

② $\sin t\, u(t-\pi)$

$F(t) = \sin t$

Soln: $F(t) = f(t-\pi)$

Doyourself

$f(t-\pi) = \sin t$

To get $f(t)$, replace t by $t+\pi$

$f(t) = \sin(t+\pi)$ → $\sin(180+\theta) = -\sin\theta$ ∵ 3rd quadrant ∵ $\sin\theta$ is -ve

$f(t) = -\sin t$

$$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t]$$

$$= -1\left[\mathcal{L}\{\sin t\}\right]$$

$$\mathcal{L}[f(t)] = \frac{-1}{s^2+1}$$

$$\mathcal{L}[f(t)] = \bar{f}(s)$$

$$\therefore \mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s) \quad (\because a=\pi)$$

$$= e^{-\pi s} \times \frac{-1}{s^2+1}$$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\bar{f}(s)$

$$\mathcal{L}[f(t-\pi)u(t-\pi)] = -\frac{e^{-\pi s}}{s^2+1} \Rightarrow \mathcal{L}[\sin t\, u(t-\pi)] = -\frac{e^{-\pi s}}{s^2+1}$$

③ $(1-e^{+2t})u(t+1)$

$F(t) = 1-e^{2t}$

Soln: $F(t) = f(t+1)$

$f(t+1) = 1-e^{+2t}$

To get f(t) replace t by t - 1

$f(t) = 1 - e^{+2(t-1)}$

$= 1 - e^{2t-2}$

$f(t) = 1 - e^{2t-2} \Rightarrow f(t) = 1 - e^{2t} \cdot e^{-2}$

$\mathcal{L}[f(t)] = \mathcal{L}[1 - e^{2t} \cdot e^{-2}]$

$= \mathcal{L}[1] - \mathcal{L}[e^{2t} \cdot e^{-2}]$

$= 1/s - e^{-2} \mathcal{L}[e^{2t}]$

$= 1/s - e^{-2} \frac{1}{s-2}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t+1)u(t+1)] = e^{s} \cdot \bar{f}(s) \quad (\because a = -1)$

$\mathcal{L}[(1-e^{2t})u(t+1)] = e^{s}\left(1/s - e^{-2}\left(\frac{1}{s-2}\right)\right)$

(4) $(3t^2 + 4t + 5)u(t-3)$

$f(t) = 3t^2 + 4t + 5$

Soln: $f(t) = f(t-3)$

$f(t-3) = 3t^2 + 4t + 5$

replace t by t + 3

$f(t) = 3(t+3)^2 + 4(t+3) + 5$

$= 3[t^2 + 9 + 6t] + 4t + 12 + 5$

$= 3t^2 + 27 + 18t + 4t + 12 + 5$

$f(t) = 3t^2 + 22t + 44$

$\mathcal{L}[f(t)] = \mathcal{L}[3t^2 + 22t + 44]$

$= 3\mathcal{L}[t^2] + 22\mathcal{L}[t] + \mathcal{L}[44]$

$= 3 \cdot \frac{2!}{s^3} + 22 \cdot \frac{1!}{s^2} + \frac{44}{s}$

w.k.t $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$\mathcal{L}[f(t)] = 6/s^3 + 22/s^2 + 44/s$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t-3)\,u(t-3)] = e^{-3s} \bar{f}(s) \quad (\because a = 3)$

$\mathcal{L}[(3t^2 + 4t + 5)\,u(t-3)] = e^{-3s}\left[6/s^3 + 22/s^2 + 44/s\right]$

(5) $(t^3 + t^2 + t + 1)\,u(t+1)$

Soln: $F(t) = t^3 + t^2 + t + 1$

$F(t) = f(t+1)$

$f(t+1) = t^3 + t^2 + t + 1$

Replace t by $t-1$

$f(t) = (t-1)^3 + (t-1)^2 + (t-1) + 1$

Do yourself

$= [\text{Do yourself} + 3t^2 + 3t] + [t^2 + \cancel{1} - 2t] + t - \cancel{1} + \cancel{1}$

$= t^3 + 2t - 2t^2 = t^3 - 2t^2 + 2t$

$\mathcal{L}[f(t)] = \mathcal{L}[t^3] - 2\mathcal{L}[t^2] + 2\mathcal{L}[t]$

$= 3!/s^4 - 2 \cdot 2!/s^3 + 2 \cdot 1/s^2$

$\mathcal{L}[f(t)] = 6/s^4 - 4/s^3 + 2/s^2 = \bar{f}(s)$

$\mathcal{L}[(t^3 + t^2 + t + 1)\,u(t+1)] = e^{s}\left[6/s^4 - 4/s^3 + 2/s^2\right]$

$(\because a = -1)$

(6) $(t^2 - 6t + 9)\,e^{-(t-3)}\,u(t-3)$

Soln: $F(t) = f(t-3)$

$f(t-3) = (t^2 - 6t + 9)\,e^{-(t-3)}$

$= (t-3)^2\,e^{-(t-3)}$

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replace t by $t+3$

$f(t) = (t)^2 e^{-t}$

$\mathcal{L}[f(t)] = \mathcal{L}[t^2 \cdot e^{-t}]$ {property ①}

$= \mathcal{L}[t^2]_{s \to s+1}$ W.K.T $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

$= \frac{2!}{(s+1)^3}$ $\mathcal{L}[t^2] = \frac{2!}{s^3}$

$= \frac{2}{(s+1)^3}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t-3)u(t-3)] = e^{-3s}\bar{f}(s)$ ($\because a = 3$)

$= e^{-3s}\left\{\frac{2}{(s+1)^3}\right\}$

$\mathcal{L}[(t^2-6t+9)e^{-(t-3)} u(t-3)] = \frac{2e^{-3s}}{(s+1)^3}$

* Express the following functions in terms of Heaviside unit step function and hence find their Laplace transform

① $f(t) = \begin{cases} t & 0 < t < 4 \\ 5 & t > 4 \end{cases}$

Soln: $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$
(by property iii)

$f(t) = t + [5-t]u(t-4)$

$\mathcal{L}[f(t)] = \mathcal{L}(t) + \mathcal{L}[(5-t)u(t-4)]$ — ①

we have $\mathcal{L}(t) = \frac{1}{s^2}$

$f(t-4) = 5-t$

Replace t by $t+4$ to get $f(t)$

$f(t) = 5-(t+4)$

$f(t) = 5-t-4$

$f(t) = 1-t$

$\mathcal{L}[f(t)] = \mathcal{L}[1] - \mathcal{L}[t]$

$\mathcal{L}[f(t)] = \frac{1}{s} - \frac{1}{s^2}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t-4)\,u(t-4)] = e^{-4s}\bar{f}(s) \quad (\because a=4)$

$\mathcal{L}[(5-t)\,u(t-4)] = e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$

use these results in ①

$\mathcal{L}[f(t)] = \frac{1}{s^2} + e^{-4s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$

$= \frac{1}{s}e^{-4s} + \frac{1}{s^2} - e^{-4s}\frac{1}{s^2}$

$\mathcal{L}[f(t)] = \frac{1}{s}e^{-4s} + \frac{1}{s^2}(1-e^{-4s})$

② $f(t) = \begin{cases} \cos t, & 0<t<\pi \\ \sin t, & t>\pi \end{cases}$

Dec 2018

Solⁿ: $f(t) = \cos t + (\sin t - \cos t)\,u(t-\pi)$ (by a property ③)

$\mathcal{L}[f(t)] = \mathcal{L}(\cos t) + \mathcal{L}[[\sin t - \cos t]\,u(t-\pi)]$ — ①

Now let $f(t-\pi) = \sin t - \cos t$

To get f(t) replace ...

$f(t) = \sin(t+\pi) - \cos(t+\pi)$

$= -\sin t - (-\cos t)$

$f(t) = -\sin t + \cos t$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t + \cos t]$

$= \mathcal{L}[-\sin t] + \mathcal{L}[\cos t]$

$= -\dfrac{1}{s^2+1} + \dfrac{s}{s^2+1}$

$= \dfrac{-1+s}{s^2+1}$

$\mathcal{L}[f(t)] = \dfrac{s-1}{s^2+1}$

$\sin(t+\pi) = -\sin t$
$t+\pi$ lies in III or
$\cos(t+\pi) = -\cos t$
$t+\pi$ lies in III or
In 3rd or sin and cos functions are $-ve$

$\mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s} F(s) \quad (\because a = \pi)$

$\mathcal{L}[(\sin t - \cos t)u(t-\pi)] = \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

put this in ①

$\mathcal{L}[f(t)] = \mathcal{L}[\cos t] + \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

$= \dfrac{s}{s^2+1} + \dfrac{e^{-\pi s}(s-1)}{s^2+1}$

$\mathcal{L}[f(t)] = \dfrac{s + e^{-\pi s}(s-1)}{s^2+1}$ //

③ June 2017

$f(t) = \begin{cases} \sin t, & 0 < t \le \pi/2 \\ \cos t, & t > \pi/2 \end{cases}$

Soln: $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$

$f(t) = \sin t + [\cos t - \sin t]u(t - \pi/2)$ (by property ③)

$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[(\cos t - \sin t)u(t - \pi/2)]$ — ①

$\mathcal{L}[\sin t] = 1/(s^2+1)$

$f(t-\pi/2) = \cos t - \sin t$

To get $f(t)$ replace t by $t+\pi/2$

$f(t) = \cos(t+\pi/2) - \sin(t+\pi/2)$

$f(t) = -\sin t - \cos t$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t - \cos t]$

$= \mathcal{L}[-\sin t] + \mathcal{L}[-\cos t]$

$= \frac{-1}{s^2+1} - \frac{s}{s^2+1}$

$= \frac{-1-s}{s^2+1}$

$= \frac{-(s+1)}{s^2+1}$

$\cos(\pi/2+t)$
$\hookrightarrow (90+\theta)$
$= -\sin t$
$\because$ $90+t$ in II or
cos function is
$-ve$

$\sin(\pi/2+t)$
$= \cos t$
$90+t$ lie in II or
sin function in $+ve$

$\mathcal{L}[f(t-\pi/2)u(t-\pi/2)] = e^{-\pi s/2}F(s)$

$\mathcal{L}[(\cos t - \sin t)u(t-\pi/2)] = e^{-\pi s/2}\frac{(-(s+1))}{s^2+1}$

use this result in (1)

$\mathcal{L}[f(t)] = \frac{1}{s^2+1} - \frac{e^{-\pi s/2}(s+1)}{s^2+1}$

$\mathcal{L}[f(t)] = \frac{1-e^{-\pi s/2}(s+1)}{s^2+1}$

(4) Do yourself $f(t) = \begin{cases} \sin 2t, & 0<t<\pi \\ 0 & t>\pi \end{cases}$

Ans: $\mathcal{L}[f(t)] = \frac{2}{s^2+4} - \frac{2e^{-\pi s}}{s^2+4}$

$= \frac{2(1-e^{-\pi s})}{s^2+4}$

(6) June 2016, 18

$$f(t) = \begin{cases} 1 & 0 < t \leq 1 \\ t & 1 < t \leq 2 \\ t^2 & t > 2 \end{cases}$$

Sol^n: by property (14)

$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) +$
$[f_3(t) - f_2(t)]u(t-b)$

$f(t) = 1 + [t-1]u(t-1) + [t^2 - t]u(t-2)$ — (A)

$f(t-1) = t-1$

$\mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[(t-1)u(t-1)] + \mathcal{L}[(t^2-t)u(t-2)]$ — (1)

w.k.T $\mathcal{L}[1] = 1/s$

$\mathcal{L}[(t-1)u(t-1)]$

$\Rightarrow F(t-1) = t-1$

To get $F(t)$ replace t by $t+1$

$F(t) = t$

$\mathcal{L}[F(t)] = \mathcal{L}[t]$

$= 1!/s^2$

$\mathcal{L}[t^n] = n!/s^{n+1}$

$\mathcal{L}[F(t-1)u(t-1)] = e^{-s}F(s)$

$\mathcal{L}[F(t)] = 1/s^2$

$\Rightarrow \mathcal{L}[(t-1)u(t-1)] = 1/s^2 \times e^{-s}$

$\mathcal{L}[F(t)] = F(s)$

$\mathcal{L}[(t^2-t)u(t-2)]$

$\Rightarrow G(t-2) = t^2 - t$

To get $G(t)$ replace t by $t+2$

$G(t) = (t+2)^2 - (t+2)$

$$= t^2 + 4 + 4t - t - 2$$

$$G(t) = t^2 + 3t + 2$$

$$\mathcal{L}[G(t)] = \mathcal{L}[t^2] + 3\mathcal{L}[t] + \mathcal{L}[2]$$

$\mathcal{L}[2]$ or $2\mathcal{L}[1] = 2/s$

$$= \frac{2!}{s^3} + 3 \cdot \frac{1!}{s^2} + 2/s$$

$$= 2/s^3 + 3/s^2 + 2/s$$

$$\mathcal{L}[G(t)] = \bar{G}(s) \Rightarrow \mathcal{L}[G(t-2)\,u(t-2)] = e^{-2s}\bar{G}(s)$$

$$\mathcal{L}[(t^2 - t)\,u(t-2)] = e^{-2s}\,\bar{G}(s)$$

$$\mathcal{L}[(t^2 - t)\,u(t-2)] = e^{-2s}\left[2/s^3 + 3/s^2 + 2/s\right]$$

put these results in (1)

$$\mathcal{L}[f(t)] = 1/s + e^{-s}/s^2 + e^{-2s}\left(2/s^3 + 3/s^2 + 2/s\right)$$

Do yourself

(5) June 2018

$$f(t) = \begin{cases} \cos t, & 0 < t \le \pi \\ 1 & \pi < t \le 2\pi \\ \sin t & t > 2\pi \end{cases}$$

Sol^n: by property (4)

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a) + [f_3(t) - f_2(t)]\,u(t-b)$$

$$f(t) = \cos t + [1 - \cos t]\,u(t-\pi) + [\sin t - 1]\,u(t-2\pi)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[\cos t] + \mathcal{L}[(1 - \cos t)\,u(t-\pi)] + \mathcal{L}[(\sin t - 1)\,u(t-2\pi)] \quad \text{— (1)}$$

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(3)

Continued

(5) $\mathcal{L}[\cos t] = \frac{s}{s^2+1}$

$\mathcal{L}[(1-\cos t)u(t-\pi)]$

$\Rightarrow$ Let $f(t-\pi) = 1-\cos t$

To get $f(t)$ replace t by $t+\pi$

$f(t) = 1-\cos(t+\pi)$

$= 1-(-\cos t))$

$f(t) = 1+\cos t$

$\cos(t+\pi) = -\cos t$

$\downarrow$

lying in III Qr

cos function is -ve in III Qr

$\mathcal{L}[f(t)] = \mathcal{L}[1+\cos t]$

$= \mathcal{L}[1] + \mathcal{L}[\cos t]$

$\mathcal{L}[f(t)] = \frac{1}{s} + \frac{s}{s^2+1}$

$\mathcal{L}[f(t)] = \bar{f}(s)$

$\mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s) \quad (\because a = \pi)$

$\mathcal{L}[(1-\cos t)u(t-\pi)] = e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right)$

$\mathcal{L}[(\sin t - 1)u(t-2\pi)]$

$\Rightarrow$ Let $G(t-2\pi) = \sin t - 1$

To get $G(t)$ replace t by $t+2\pi$

$G(t) = \sin(t+2\pi) - 1$

$G(t) = \sin t - 1$

$\sin(2\pi + t)$

$= \sin t$

$2\pi + t$ lies in I Qr

In I Qr all trigonometric functions are +ve

$\mathcal{L}[G(t)] = \mathcal{L}[\sin t - 1]$

$= \mathcal{L}[\sin t] - \mathcal{L}[1]$

$\mathcal{L}[G(t)] = \frac{1}{s^2+1} - \frac{1}{s}$

$\mathcal{L}[G(t)] = \bar{G}(s)$

$\mathcal{L}[G_1(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\,\bar{G}_1(s)$

$\mathcal{L}[(\sin t - 1)\,u(t-2\pi)] = e^{-2\pi s}\left[\frac{1}{s^2+1} - \frac{1}{s}\right]$

put all these results in ①

$$\mathcal{L}[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s}\right)$$

Do yourself

* $f(t) = \begin{cases} \cos t, & 0<t<2\pi \\ \cos 2t, & \pi<t<2\pi \\ \cos 3t, & t>2\pi \end{cases}$ (use property (iv))

Ans: $\mathcal{L}[f(t)] = \frac{s}{s^2+1} + s e^{-\pi s}\left(\frac{1}{s^2+4} + \frac{1}{s^2+1}\right) - \frac{5 s e^{-2\pi s}}{(s^2+4)(s^2+9)}$

* $f(t) = \begin{cases} e^{2t} & 0<t<1 \\ 2, & t>1 \end{cases}$

(use property (iii))

Ans: $\mathcal{L}[f(t)] = \frac{1}{s-2} + e^{-s}\left(\frac{2}{s} - \frac{e^2}{s-2}\right)$

⑧ Express the function $f(t) = \begin{cases} \pi - t, & 0<t\le\pi \\ \sin t, & t>\pi \end{cases}$ interms of unit step function and hence find its laplace transform

Dec 2017

Soln: by the property of unit step function

$f(t) = f_1(t) + [f_2(t) - f_1(t)\,u(t-a)]$

$f(t) = \pi - t + [\sin t - (\pi - t)]\,u(t-\pi)]$

$\mathcal{L}[f(t)] = \mathcal{L}[\pi - t] + \mathcal{L}[(\sin t - \pi + t)\,u(t-\pi)]$ — ①

$\mathcal{L}[\pi - t] = \mathcal{L}[\pi] - \mathcal{L}[t]$

$= \pi\,\mathcal{L}[1] - \mathcal{L}[t]$

$= \pi \cdot \frac{1}{s} - \frac{1!}{s^2}$ $\left(\because \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}\right)$

$\mathcal{L}[\pi - t] = \frac{\pi}{s} - \frac{1}{s^2}$

$f(t-\pi) = \sin t - \pi + t$

to get $f(t)$ replace t by $t+\pi$

$f(t) = \sin(t+\pi) - \cancel{\pi} + t + \cancel{\pi}$

$f(t) = -\sin t + t$

$\mathcal{L}[f(t)] = \mathcal{L}[-\sin t + t]$

$= \mathcal{L}[-\sin t] + \mathcal{L}[t]$

$= \dfrac{-1}{s^2+1} + \dfrac{1!}{s^2}$

$\mathcal{L}[f(t)] = \dfrac{-1}{s^2+1} + \dfrac{1}{s^2}$

$\mathcal{L}[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s)$

$\mathcal{L}[(\sin t - \pi + t)u(t-\pi)] = e^{-\pi s}\left(\dfrac{-1}{s^2+1} + \dfrac{1}{s^2}\right)$

put thye all in 1

$\mathcal{L}[f(t)] = \left(\dfrac{\pi}{s} - \dfrac{1}{s^2}\right) + e^{-\pi s}\left(\dfrac{-1}{s^2+1} + \dfrac{1}{s^2}\right)$

$= \left(\dfrac{\pi}{s} - \dfrac{1}{s^2}\right) + e^{-\pi s}\left(\dfrac{-\cancel{s^2} + \cancel{s^2} + 1}{s^2(s^2+1)}\right)$

$\mathcal{L}[f(t)] = \left(\dfrac{\pi}{s} - \dfrac{1}{s^2}\right) + e^{-\pi s}\left(\dfrac{1}{s^2(s^2+1)}\right)$

(2) Define heaviside unit step function. Using unit step function find Laplace transform of $f(t) = \begin{cases} \sin t & 0 \le t < \pi \\ \sin 2t & \pi \le t < 2\pi \\ \sin 3t & t \ge 2\pi \end{cases}$

Dec 2016

Sol^n:

write the definition of unit step function

by using the standard property

$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$

$f(t) = \sin t + [(\sin 2t - \sin t)]u(t-\pi) + [\sin 3t - \sin 2t]u(t-2\pi)$

$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[(\sin 2t - \sin t)u(t-\pi)] + \mathcal{L}[(\sin 3t - \sin 2t)u(t-2\pi)]$ — (1)

$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$

Let $F(t-\pi) = \sin 2t - \sin t$

to get $F(t)$ replace t by $t+\pi$

$F(t) = \sin 2(t+\pi) - \sin(t+\pi)$

$= \sin(2t+2\pi) - \sin(t+\pi)$

$360+2t$ i.e. $(360+\theta)$ → I quadrant; $180+t$ → III quadrant

$= \sin 2t - (-\sin t)$

$F(t) = \sin 2t + \sin t$

$\mathcal{L}[F(t)] = \mathcal{L}[\sin 2t] + \mathcal{L}[\sin t]$

$= \frac{2}{s^2+4} + \frac{1}{s^2+1}$

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$$\mathcal{L}[F(t)] = \bar{F}(s)$$

$$\mathcal{L}[F(t-\pi)\, u(t-\pi)] = e^{-\pi s}\bar{F}(s)$$

$$\mathcal{L}\{[\sin 2t - \sin t]\, u(t-\pi)\} = e^{-\pi s}\left\{\frac{2}{s^2+4} + \frac{1}{s^2+1}\right\}$$

Let $G(t-2\pi) = \sin 3t - \sin 2t$

To get $G(t)$ replace t by $t+2\pi$

$$G(t) = \sin 3(t+2\pi) - \sin 2(t+2\pi)$$

$$= \sin(3t+6\pi) - \sin(2t+2\pi)$$

$\sin(360+2t)$

$= \sin 2t$

$\sin(3(2\pi)+3t)$

$= \sin 3t$

$$G(t) = \sin 3t - \sin 2t$$

$$\mathcal{L}[G(t)] = \mathcal{L}[\sin 3t - \sin 2t]$$

$$= \mathcal{L}[\sin 3t] - \mathcal{L}[\sin 2t]$$

$$\mathcal{L}[G(t)] = \frac{3}{s^2+9} - \frac{2}{s^2+4}$$

$$\mathcal{L}[G(t)] = \bar{G}(s)$$

$$\mathcal{L}[G(t-2\pi)\, u(t-2\pi)] = e^{-2\pi s}\bar{G}(s)$$

$$\mathcal{L}[(\sin 3t - \sin 2t)\, u(t-2\pi)] = e^{-2\pi s}\left[\frac{3}{s^2+9} - \frac{2}{s^2+4}\right]$$

Inverse Laplace Transforms

If $\mathcal{L}[f(t)] = \bar{f}(s)$, then $f(t)$ is called inverse Laplace transform of $\bar{f}(s)$ & is denoted by $\mathcal{L}^{-1}[\bar{f}(s)]$

Thus we can say that

$$\mathcal{L}[f(t)] = \bar{f}(s) \Leftrightarrow \mathcal{L}^{-1}[\bar{f}(s)] = f(t)$$

eg: $\mathcal{L}(1) = \frac{1}{s} \Rightarrow \mathcal{L}^{-1}(\frac{1}{s}) = 1$

$\mathcal{L}(\cos at) = \frac{s}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}(\frac{s}{s^2+a^2}) = \cos at$

NOTE:
1. $\mathcal{L}^{-1}[\frac{1}{s}] = 1$
2. $\mathcal{L}^{-1}[\frac{1}{s-a}] = e^{at}$
3. $\mathcal{L}^{-1}[\frac{1}{s+a}] = e^{-at}$.
4. $\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$
5. $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
6. $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
7. $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$
8. $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$
9. $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$
10. $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] (n>-1) = \frac{t^n}{\Gamma(n+1)}$
11. $\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right]$ $n=1,2,3\ldots = \frac{t^n}{n!}$

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Note: ① $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^{t}$

② $\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}$

③ $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] = \cos 3t$

④ $\mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$\mathcal{L}^{-1}\left[\frac{3}{s^2+9}\right] = \sin 3t$

⑤ $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$\mathcal{L}^{-1}\left[\frac{s}{s^2-16}\right] = \cosh 4t$

⑥ $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2+5}\right] = \frac{1}{\sqrt{5}} \sin(\sqrt{5}\,t)$

$(\sqrt{5})^2$

⑦ $\mathcal{L}^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$

$\mathcal{L}^{-1}\left[\frac{4}{s^2-16}\right] = \sinh 4t$

⑧ $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2-36}\right] = \frac{\sinh 6t}{6}$

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(9) $\mathcal{L}^{-1}\left(\frac{1}{s^{3/2}}\right) = \frac{t^{1/2}}{\Gamma(3/2)} = \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} = 2\sqrt{t/\pi}$

(10) $\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{t^3}{3!}$

$\mathcal{L}^{-1}[c_1\bar{f}(s) + c_2\bar{g}(s)] = c_1\mathcal{L}^{-1}[\bar{f}(s)] + c_2\mathcal{L}^{-1}[\bar{g}(s)]$

Find the inverse Laplace transform of the following

(1) $\frac{1}{s+2} + \frac{3}{2s+5} - \frac{4}{3s-2}$

Soln: $= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + \mathcal{L}^{-1}\left[\frac{3}{2s+5}\right] - \mathcal{L}^{-1}\left[\frac{4}{3s-2}\right]$

$= \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{2s+5}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{3s-2}\right]$

$= e^{-2t} + 3\mathcal{L}^{-1}\left[\frac{1}{2(s+(5/2))}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{3(s-2/3)}\right]$

$= e^{-2t} + \frac{3}{2}\mathcal{L}^{-1}\left[\frac{1}{s+5/2}\right] - \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s-2/3}\right]$

$= e^{-2t} + \frac{3}{2}e^{-5/2\,t} - \frac{4}{3}e^{2/3\,t}$

(2) $\frac{2s-5}{4s^2+25} + \frac{8-6s}{16s^2+9}$

Soln: $\frac{2s}{4s^2+25} - \frac{5}{4s^2+25} + \frac{8}{16s^2+9} - \frac{6s}{16s^2+9}$

$2\mathcal{L}^{-1}\left[\frac{s}{4s^2+25}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{4s^2+25}\right] + 8\mathcal{L}^{-1}\left[\frac{1}{16s^2+9}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{16s^2+9}\right]$

$2\mathcal{L}^{-1}\left[\frac{s}{4(s^2+\frac{25}{4})}\right] - 5\mathcal{L}^{-1}\left[\frac{1}{4(s^2+\frac{25}{4})}\right] + 8\mathcal{L}^{-1}\left[\frac{1}{16(s^2+9/16)}\right] - 6\mathcal{L}^{-1}\left[\frac{s}{16(s^2+\frac{9}{16})}\right]$

$\frac{2}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2+(\frac{5}{2})^2}\right] - \frac{5}{4}\mathcal{L}^{-1}\left[\frac{1}{s^2+(\frac{5}{2})^2}\right] + \frac{8}{16}\mathcal{L}^{-1}\left[\frac{1}{s^2+(3/4)^2}\right] - \frac{6}{16}\mathcal{L}^{-1}\left[\frac{s}{(s^2+(\frac{3}{4})^2)}\right]$

$\frac{1}{2}\cos(5/2)t - \frac{5}{4}\cdot\frac{\sin(5/2)t}{5/2} + \frac{1}{2}\frac{\sin(3/4)t}{3/4}$

$-\frac{3}{8}\times\cos(3/4)t$

$\frac{1}{2}\cos(5/2)t - \frac{2}{4}\sin(5/2)t + \frac{1}{2}\times\frac{4}{3}\sin(3/4)t - \frac{3}{8}\cos(3/4)t$

$\frac{1}{2}\cos(5/2)t - \frac{1}{2}\sin(5/2)t + \frac{2}{3}\sin(3/4)t - \frac{3}{8}\cos(3/4)t$

$\frac{1}{2}\cos(5/2)t - \frac{1}{2}\sin(5/2)t$

Doyourself

(3) $\frac{s+2}{s^2+36} + \frac{4s-1}{s^2+25}$ Ans: $\cos 6t + \frac{1}{3}\sin 6t + 4\cos 5t - \frac{1}{5}\sin 5t$

(4) $\frac{2s-5}{8s^2-50} + \frac{4s}{9-s^2}$ Ans: $\frac{1}{4}\cdot e^{-5t/2} - 4\cosh 3t$

Ans: $= \frac{2s}{8s^2-50} - \frac{5}{8s^2-50} + \frac{4s}{9-s^2}$

$= \frac{2s}{2(4s^2-25)} - \frac{5}{2(4s^2-25)} + \frac{4s}{9-s^2}$

$= \frac{s}{4(s^2-25/4)} - \frac{5}{2\times4(s^2-\frac{25}{4})} - \frac{4s}{s^2-9}$

$= \frac{s}{4(s^2-(5/2)^2)} - \frac{5}{8(s^2-(5/2)^2)} - \frac{4s}{s^2-9}$

$= \frac{1}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2-(5/2)^2}\right] - \frac{5}{8}\mathcal{L}^{-1}\left[\frac{1}{s^2-(5/2)^2}\right] - 4\mathcal{L}^{-1}\left[\frac{s}{s^2-9}\right]$

$= \frac{1}{4}\cosh\frac{5t}{2} - \frac{5}{8}\frac{\sinh\frac{5t}{2}}{5/2} - 4\cosh 3t$

$= \frac{1}{4}\cosh\frac{5}{2}t - \frac{1}{4}\sinh\frac{5}{2}t - 4\cosh 3t$

$= \frac{1}{4}\left\{\frac{e^{5/2 t}+e^{-5/2 t}}{2} - \frac{e^{5/2 t}-e^{-5/2 t}}{2}\right\} - 4\cosh 3t$

$= \frac{1}{4}\left\{\frac{2e^{-5/2 t}}{2}\right\} - 4\cosh 3t$

$= \frac{1}{4}e^{-5/2 t} - 4\cosh 3t$

(4) $\frac{2s-5}{8s^2-50} + \frac{4s}{9-s^2}$

$= \mathcal{L}^{-1}\left[\frac{2s-5}{2(4s^2-25)}\right] + 4\mathcal{L}^{-1}\left[\frac{s}{9-s^2}\right]$

$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2s-5}{(2s)^2-5^2}\right] - 4\mathcal{L}^{-1}\left[\frac{s}{s^2-9}\right]$

$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{\cancel{2s-5}}{(2s+5)\cancel{(2s-5)}}\right] - 4\cosh 3t$

$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{2s+5}\right] - 4\cosh 3t$

$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{2(s+5/2)}\right] - 4\cosh 3t$

$= \frac{1}{2}\cdot\frac{1}{2}\cdot e^{-\frac{5t}{2}} - 4\cosh 3t$

$= \frac{1}{4}e^{-5t/2} - 4\cosh 3t$

(5) $\frac{(s+2)^3}{s^6}$

Solⁿ: we $(a+b)^3 = a^3+b^3+3a^2b+3ab^2$

$(s+2)^3 = s^3+8+6s^2+12s$

$$\frac{(s+2)^3}{s^6} = \frac{s^3+8+6s^2+12s}{s^6}$$

$$= \frac{1}{s^3} + \frac{8}{s^6} + \frac{6}{s^4} + \frac{12}{s^5}$$

$$= L^{-1}\left[\frac{1}{s^3}\right] + L^{-1}\left[\frac{8}{s^6}\right] + L^{-1}\left[\frac{6}{s^4}\right] + L^{-1}\left[\frac{12}{s^5}\right]$$

$$= L^{-1}\left[\frac{1}{s^3}\right] + 8L^{-1}\left[\frac{1}{s^6}\right] + 6L^{-1}\left[\frac{1}{s^4}\right] + 12L^{-1}\left[\frac{1}{s^5}\right]$$

$$= \frac{t^2}{2!} + 8 \cdot \frac{t^5}{5!} + 6 \cdot \frac{t^3}{3!} + 12 \cdot \frac{t^4}{4!}$$

$$= \frac{t^2}{2} + 8 \cdot \frac{t^5}{120} + 6\frac{t^3}{6} + 12 \cdot \frac{t^4}{24}$$

$$L^{-1}\left[\frac{(s+2)^3}{s^6}\right] = \frac{t^2}{2} + \frac{t^5}{15} + t^3 + \frac{t^4}{2}$$

* Do yourself

* $\frac{3(s^2-1)^2}{2s^5}$

Ans: $\frac{3}{2}\left[1 - t^2 + \frac{t^4}{24}\right]$

Computation of the inverse transform of $\bar{e}^{as}\bar{f}(s)$

W.K.T $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\mathcal{L}^{-1}[e^{-as}\bar{f}(s)] = f(t-a)u(t-a)$$

Working procedure:

① In the given function we should observe the presence of e^{-as} first and identify the remaining part of the function to be called as $\bar{f}(s)$.

② Taking the inverse of $\bar{f}(s)$ we obtain $f(t)$

③ The required inverse of $e^{-as}\bar{f}(s)$ is obtained by replacing t by $t-a$ in $f(t)$ to be multiplied by unit step function $u(t-a)$.

(*) Find the inverse Laplace transform of the following

① $\dfrac{1+e^{-3s}}{s^2}$

Soln:

$$\mathcal{L}^{-1}\left[\frac{1+e^{-3s}}{s^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^2}\right] = (t-3)u(t-3)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

$$= t + (t-3)u(t-3)$$

$$\therefore \mathcal{L}^{-1}\left[\frac{1+e^{-3s}}{s^2}\right] = t + (t-3)u(t-3)$$

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(2) $\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}$

Sol^n: $\mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$

$= \mathcal{L}^{-1}\left[\frac{3}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{2e^{-s}}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{3e^{-2s}}{s}\right]$

$= 3\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 2\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^3}\right] - 3\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s}\right]$ —— (1)

$\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = \frac{t}{1!} = t$

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{1+1}}\right] = \frac{t^1}{1!} = t$

$\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!} = \frac{t^2}{2}$

$\therefore n = 1$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

$\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{2+1}}\right] = \frac{t^2}{2}$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{0+1}}\right] = \frac{t^0}{0!} =$

hence (1) becomes

$\mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right]$

$= 3t + 2\cdot\frac{(t-1)^2}{2}u(t-1) - 3(1)u(t-2)$

$\mathcal{L}^{-1}\left[\frac{3}{s^2} + \frac{2e^{-s}}{s^3} - \frac{3e^{-2s}}{s}\right] = 3t + (t-1)^2 u(t-1) - 3u(t-2)$

(3) Find the inverse Laplace transform of

$\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}$

Sol^n: $\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right] + \mathcal{L}^{-1}\left[\frac{se^{-2\pi s}}{s^2+4}\right]$ —(1)

$\mathcal{L}[\sin t] = \frac{1}{s^2+1}$

$\mathcal{L}[\cos 2t] = \frac{s}{s^2+4}$

W.K.T. $\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$

$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$

① becomes

$= \sin(t-\pi)u(t-\pi) + \cos 2(t-2\pi)u(t-2\pi)$

$$\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2+1} + \frac{se^{-2\pi s}}{s^2+1}\right] = -\sin t\, u(t-\pi) + \cos 2t\, u(t-2\pi)$$

$\sin(t-\pi)$
$= \sin(-(\pi-t))$
w.k.T $\sin(-\theta) = -\sin\theta$
$= -\sin(\pi-t)$ → 2nd Q
$= -\sin t$

* Doyourself

④ $\dfrac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}$

Ans: $\mathcal{L}^{-1}\left[\dfrac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right] = \sin\pi t\, u(t-\tfrac{1}{2}) - \sin\pi t\, u(t-1)$

common

$$\mathcal{L}^{-1}\left[\frac{se^{-s/2} + \pi e^{-s}}{s^2+\pi^2}\right] = \sin\pi t\left[u(t-\tfrac{1}{2}) - u(t-1)\right]$$

⑤ $\dfrac{\cosh 2s}{e^{3s}\cdot s^2}$

Solⁿ: $\dfrac{\cosh 2s}{e^{3s}\cdot s^2} = \dfrac{\cosh 2s}{s^2}\times e^{-3s}$

$$= \frac{e^{-3s}}{s^2}\left[\frac{e^{2s}+e^{-2s}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-s}+e^{-5s}}{s^2}\right]$$

$$\frac{\cosh 2s}{e^{3s}\cdot s^2} = \frac{1}{2}\left[\frac{e^{-s}}{s^2} + \frac{e^{-5s}}{s^2}\right]$$

$$\mathcal{L}^{-1}\left[\frac{\cosh 2s}{e^{3s}\cdot s^2}\right] = \frac{1}{2}\left\{\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-5s}}{s^2}\right]\right\}$$

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W.K.T $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$

$$\mathcal{L}^{-1}\left[\frac{\cosh 2s}{e^{3s}\cdot s^2}\right] = \frac{1}{2}\left\{(t-1)u(t-1) + (t-5)u(t-5)\right\}$$

(6) Do yourself

$$\frac{(1-e^{-s})(2-e^{-2s})}{s^3}$$

Soln: $$\frac{(1-e^{-s})(2-e^{-2s})}{s^3} = \frac{2-e^{-2s}-2e^{-s}+e^{-s}\cdot e^{-2s}}{s^3}$$

$$= \frac{2-e^{-2s}-2e^{-s}+e^{-3s}}{s^3}$$

$$\mathcal{L}^{-1}\left[\frac{(1-e^{-s})(2-e^{-2s})}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{2-e^{-2s}-2e^{-s}+e^{-3s}}{s^3}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{2e^{-s}}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^3}\right]$$

$$= 2\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^3}\right] - 2\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^3}\right] + \mathcal{L}^{-1}\left[\frac{e^{-3s}}{s^3}\right]$$

$$= 2\cdot\frac{t^2}{2!} - \frac{(t-2)^2}{2!}u(t-2) - 2\frac{(t-1)^2}{2!}u(t-1) + \frac{(t-3)^2}{2!}u(t-3)$$

$$= t^2 - \frac{(t-2)^2u(t-2)}{2} - (t-1)^2u(t-1) + \frac{(t-3)^2u(t-3)}{2}$$

$$= t^2 - (t-1)^2u(t-1) - \frac{(t-2)^2u(t-2)}{2} + \frac{(t-3)^2u(t-3)}{2}$$

** Inverse transform by completing the Square

we have property that if $\mathcal{L}[f(t)] = \bar{f}(s)$ then $\mathcal{L}[e^{at} f(t)] = \bar{f}(s-a) \Rightarrow \mathcal{L}^{-1}[\bar{f}(s)] = f(t)$ - ①

$\Rightarrow \mathcal{L}^{-1}[\bar{f}(s-a)] = e^{at} f(t)$ - ②

from ① and ② we have

$\mathcal{L}^{-1}[\bar{f}(s-a)] = e^{at} \mathcal{L}^{-1}[\bar{f}(s)]$ - ③

working procedure

① Given $\bar{f}(s) = \dfrac{\phi(s)}{ps^2+qs+r}$

we first express (ps^2+qs+r) in the form $(s-a)^2 \pm b^2$ and later express $\phi(s)$ in terms of $(s-a)$ so that given function of s reduces to a function $s-a$

② we use ③ to obtain the result

③ However if $\bar{f}(s) = \phi(s)/\varphi(s-a)$ we only need to express $\phi(s)$ in terms of $(s-a)$ to find inverse transform.

* find the inverse Laplace transform of the following functions

① $\dfrac{s+5}{s^2-6s+13}$

Soln: $\mathcal{L}^{-1}\left[\dfrac{s+5}{s^2-6s+13}\right] = \mathcal{L}^{-1}\left[\dfrac{s+5}{s^2-2\times 3s+13}\right]$

$a=s \quad b=3$

$$= \mathcal{L}^{-1}\left[\frac{s+5+3-3}{s^2-6s+9-9+13}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3+5+3}{(s-3)^2-9+13}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3+8}{(s-3)^2+4}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{(s-3)+8}{(s-3)^2+2^2}\right]$$

here $a=3$ and $s-3$ change to s

$$= e^{3t}\,\mathcal{L}^{-1}\left[\frac{s+8}{s^2+2^2}\right]$$

$$= e^{3t}\left\{\mathcal{L}^{-1}\left[\frac{s}{s^2+2^2}\right] + \mathcal{L}^{-1}\left[\frac{8}{s^2+2^2}\right]\right\}$$

$$= e^{3t}\left\{\cos 2t + 4\,\mathcal{L}^{-1}\left[\frac{2}{s^2+2^2}\right]\right\}$$

$$= e^{3t}\left\{\cos 2t + 4\cdot\sin 2t\right\}$$

$$\mathcal{L}^{-1}\left[\frac{s+5}{s^2-6s+13}\right] = e^{3t}\left\{\cos 2t + 4\sin 2t\right\}$$

② $\cancel{\dfrac{(s+2)e^{-s}}{(s+1)^4}}$

Sol^n: $\cancel{\mathcal{L}^{-1}\left[\dfrac{(s+2)e^{-s}}{(s+1)^4}\right]} =$

(2) $\dfrac{s+1}{s^2+6s+9}$

Sol^n: $L^{-1}\left[\dfrac{s+1}{s^2+6s+9}\right] = L^{-1}\left[\dfrac{(s+3)-2}{(s+3)^2}\right]$

here $a=-3$ & $s-3$ changes to s

$$= e^{-3t} L^{-1}\left[\frac{s-2}{s^2}\right]$$

$$= e^{-3t}\left\{L^{-1}\left[\frac{s}{s^2}\right] - 2L^{-1}\left[\frac{1}{s^2}\right]\right\}$$

$$= e^{-3t}\left\{L^{-1}\left[1/s\right] - 2L^{-1}\left[1/s^2\right]\right\}$$

$$= e^{-3t}\left\{1 - 2\cdot t\right\}$$

$$L^{-1}\left[\frac{s+1}{s^2+6s+9}\right] = e^{-3t}(1-2t)$$

(3) $\dfrac{e^{-4s}}{(s-4)^2}$

Sol^n: Let $\bar{f}(s) = \dfrac{1}{(s-4)^2}$

$L^{-1}\left[\bar{f}(s)\right] = L^{-1}\left[\dfrac{1}{(s-4)^2}\right]$ $\therefore a=4$ and $s-4$ changes to s

$$= e^{4t} L^{-1}(1/s^2)$$

$$= e^{4t}\cdot t$$

$$L^{-1}\left[\bar{f}(s)\right] = f(t)$$

$$L^{-1}\left[\frac{e^{-4s}}{(s-4)^2}\right] = f(t-4)u(t-4)$$

$$L^{-1}\left[\frac{e^{-4s}}{(s-4)^2}\right] = \left\{e^{4(t-4)}(t-4)\right\}u(t-4)$$

(a) $\frac{(s+2)e^{-s}}{(s+1)^4}$

Sol^n: $\bar{f}(s) = \frac{s+2}{(s+1)^4}$

we shall first find $L^{-1}[\bar{f}(s)] = f(t)$

$$L^{-1}\left[\frac{s+2}{(s+1)^4}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^4}\right]$$

here $a = -1$, $s+1$ changes to s

$$= e^{-t} L^{-1}\left[\frac{s+1}{s^4}\right]$$

$$= e^{-t}\left\{L^{-1}\left(\frac{s}{s^4}\right) + L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}\left\{L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s^4}\right)\right\}$$

$$= e^{-t}\left\{\frac{t^2}{2!} + \frac{t^3}{3!}\right\}$$

$$= e^{-t}\left\{\frac{t^2}{2} + \frac{t^3}{6}\right\}$$

$$L^{-1}[e^{-s}\bar{f}(s)] = f(t-1)u(t-1)$$

$$L^{-1}\left[e^{-s}\frac{s+2}{(s+1)^4}\right] = e^{-(t-1)}\left\{\frac{(t-1)^2}{2} + \frac{(t-1)^3}{6}\right\}u(t-1)$$

Inverse transform by the method of partial fractions

W.K.T the method of partial fraction is a technique of converting an algebraic function $\phi(s)$ in to a sum. $\psi(s)$

Depending on the nature of terms in $\psi(s)$ we have to split into a sum of various terms with constants A, B, C, D... which can be determined. Later the inverse is found term by term.

* $$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

* $$\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

* $$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

* Find the inverse laplace transform of the following functions

(1) $$\frac{1}{s(s+1)(s+2)(s+3)}$$

Sol^n:

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3} \quad \text{--- (1)}$$

$\times^{ly}$ B.S by $s(s+1)(s+2)(s+3)$

$$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + D(s)(s+1)(s+2)$$

put $s=0$

$1 = A(1)(2)(3) + 0 + 0 + 0$

$1 = 6A$

$A = 1/6$

put $s = -1$

$1 = A(0)(1)(2) + B(-1)(1)(2) + C(-1)(0)(2) + D(-1)(0)(1)$

$1 = 0 - 2B + 0 + 0$

$1 = -2B$

$B = -1/2$

put $s = -2$

$1 = A(0) + B(0) + C(-2)(-1)(1) + D(0)$

$1 = 0 + 0 + 2C \pm 0$

$1 = 2C$

$C = 1/2$

put $s = -3$

$1 = A(0) + B(0) + C(0) + D(-3)(-2)(-1)$

$1 = 0 + 0 + 0 - 6D$

$1 = -6D$

$D = -1/6$

(1) $\Rightarrow$

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{1/6}{s} + \frac{-1/2}{s+1} + \frac{1/2}{s+2} + \frac{-1/6}{s+3}$$

$$\frac{1}{s(s+1)(s+2)(s+3)} = \frac{1}{6s} - \frac{1}{2(s+1)} + \frac{1}{2(s+2)} - \frac{1}{6(s+3)}$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{6}L^{-1}\left[\frac{1}{s+3}\right]$$

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$$L^{-1}\left[\frac{1}{s(s+1)(s+2)(s+3)}\right] = \frac{1}{6} - \frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t} - \frac{1}{6}e^{-3t}$$

(2) Find $L^{-1}\left[\frac{3s+2}{s^2-s-2}\right]$

$s^2 - s - 2$

$s^2 - 2s + s - 2$

$s(s-2) + 1(s-2)$

$(s-2)(s+1)$

Sol: $\frac{3s+2}{s^2-s-2} = \frac{3s+2}{(s-2)(s+1)}$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1} \quad -(1)$$

$\times^{ly}$ B.S by $(s-2)(s+1)$

$$3s+2 = A(s+1) + B(s-2)$$

put $s = 2$

$$3(2) + 2 = A(2+1) + B(0)$$

$$8 = 3A$$

$$A = 8/3$$

put $s = -1$

$$3(-1) + 2 = A(0) + B(-3)$$

$$+1 = +3B$$

$$B = 1/3$$

put A and B in (1)

$$\frac{3s+2}{(s-2)(s+1)} = \frac{8/3}{s-2} + \frac{1/3}{s+1}$$

$$\frac{3s+2}{(s-2)(s+1)} = \frac{8}{3(s-2)} + \frac{1}{3(s+1)}$$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}\left[\frac{3s+2}{(s-2)(s+1)}\right] = \frac{8}{3}e^{2t} + \frac{1}{3}e^{-t}$$

3) $\frac{s+2}{s^2(s+3)}$

Sol: $\frac{s+2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$

$\times^{ly}$ B.S by $s^2(s+3)$

$s+2 = As(s+3) + B(s+3) + Cs^2$ — ①

$s+2 = A(s^2+3s) + B(s+3) + Cs^2$

co-eff of s^2 is A

put $s=0$

$2 = 0 + B(3) + 0$

$3B = 2$

$B = 2/3$

put $s=-3$

$-3+2 = As(0) + B(0) + 9C$

$-1 = 0+0+9C$

$9C = -1$

$C = -1/9$

equating the co-efficient of s^2 on both sidy of ①

$0 = A + C$

$A = -C$

$A = -(-1/9)$

$A = 1/9$

put A, B, C values in ①

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1/9}{s} + \frac{2/3}{s^2} + \frac{-1/9}{s+3}$$

$$\frac{s+2}{s^2(s+3)} = \frac{1}{9s} + \frac{2}{3s^2} - \frac{1}{9(s+3)}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9} L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9} L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{9}(1) + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}(1 - e^{-3t}) + \frac{2}{3}t$$

(4) $$\frac{4s+5}{(s+1)^2(s+2)}$$

Sol^n: $$\frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2} \quad \text{— (1)}$$

x^{ly} B.S by $(s+1)^2(s+2)$

$$4s+5 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \quad \text{— (*)}$$

put $s=-1$

$$4(-1)+5 = A(0) + B(-1+2) + C(0)$$

$$1 = 0 + B + 0$$

$$\therefore B = 1$$

put $s=-2$

$$4(-2)+5 = A(-2+1)(0) + B(0) + C(-2+1)^2$$

$$-3 = 0 + 0 + C(-1)^2$$

$$-3 = C$$

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$\therefore \underline{C = -3}$

Equating the Co-efficient of S^2 on both sidy of ⊛ we get,

$$0 = A + C$$
$$0 = A - 3$$
$$\underline{A = 3}$$

put A. B. C valuey in ①

$$\frac{4S+5}{(S+1)^2(S+2)} = \frac{3}{S+1} + \frac{1}{(S+1)^2} + \frac{-3}{S+2}$$

$$\mathcal{L}^{-1}\left[\frac{4S+5}{(S+1)^2(S+2)}\right] = 3\mathcal{L}^{-1}\left[\frac{1}{S+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{(S+1)^2}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{S+2}\right]$$

$$= 3e^{-t} + e^{-t}\mathcal{L}^{-1}[1/S^2] - 3e^{-2t}$$

$$\mathcal{L}^{-1}\left[\frac{4S+5}{(S+1)^2(S+2)}\right] = \underline{3e^{-t} + e^{-t}.t - 3e^{-2t}}$$

⑧ $\frac{S+2}{S^2(S+3)}$

Soln: $\frac{S+2}{S^2(S+3)} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S+3}$ — ①

$\times^{ly}$ B.S by $S^2(S+3)$

$$S+2 = AS(S+3) + B(S+3) + CS^2 \quad — ⊛$$

put $S = 0$

$2 = 0 + 3B + 0 \Rightarrow 3B = 2 \Rightarrow B = 2/3$

put $S = -3$

$$-3+2 = A(0) + 0 + C(-3)^2$$
$$-1 = 9C$$
$$\underline{C = -1/9}$$

equating the Co-efficient of s^2 on both side of ⊛ we get

$0 = A + C$

$\therefore A = -C = -(-1/9)$

$\underline{A = 1/9}$

put A, B, C values in ①

$$\frac{s+2}{s^2(s+3)} = \frac{1/9}{s} + \frac{2/3}{s^2} + \frac{-1/9}{s+3}$$

$$\frac{s+2}{s^2(s+3)} = \frac{1}{9s} + \frac{2}{3}\cdot\frac{1}{s^2} + \frac{-1}{9(s+3)}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{1}{9}(1) + \frac{2}{3}\cdot t - \frac{1}{9}e^{-3t}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$ //

⑥ $\dfrac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}$

$\underline{Sol^n}$: $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$ — ①

x^{ly} B.S by $(s-1)(s^2+1)$

$3s+1 = A(s^2+1) + (Bs+C)(s-1)$ — ⊛

put $s = 1$; $4 = A(1+1) + (B(1)+C)(0)$

$4 = 2A + 0$

$2A = 4$

$\underline{A = 2}$

put $s=0$

$1 = A(1) + (B(0)+C)(0-1)$

$1 = A + (0+C)(-1)$

$1 = A + (C)(-1)$

$1 = A - C$

W.K.T $A = 2$

$1 = 2 - C$

$1 - 2 = -C$

$-1 = -C$

$C = 1$

equating the co-efficient of s^2 on B.S of (1) we get

$0 = A + B$

$\therefore B = -A$

$B = -2$

(1) $\Rightarrow$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

$$\frac{3s+1}{(s-1)(s^2+1)} = \left[\frac{2}{s-1}\right] + \left[\frac{-2s}{s^2+1}\right] + \left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2L^{-1}\left[\frac{1}{s-1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{3s+1}{(s-1)(s^2+1)}\right] = 2e^{t} - 2\cos t + \sin t$$

$$L^{-1}\left[\frac{(3s+1)e^{-3s}}{(s-1)(s^2+1)}\right] = \left[2e^{t-3} - 2\cos(t-3) + \sin(t-3)\right]u(t-3)$$

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* Inverse transform of logarithmic functions and inverse functions

Given $\bar{f}(s)$ we need to find $L^{-1}[\bar{f}(s)] = f(t)$

we have the property $L[tf(t)] = -\bar{f}'(s)$ — (1)

equivalently, $L^{-1}[-\bar{f}'(s)] = tf(t)$.

* Find the inverse laplace transform of the following functions:-

(1) $\log\left[\frac{s+a}{s+b}\right]$

Solⁿ: let $\bar{f}(s) = \log\left[\frac{s+a}{s+b}\right]$

$\log(m/n) = \log m - \log n$

$\log(mn) = \log m + \log n$

$\bar{f}(s) = \log(s+a) - \log(s+b)$

D.w.r. to s

$$\bar{f}'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

x'ly B.S by -1

$$-\bar{f}'(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$-\bar{f}'(s) = \frac{1}{s+b} - \frac{1}{s+a}$$

$$L^{-1}[-\bar{f}'(s)] = L^{-1}\left[\frac{1}{s+b} - \frac{1}{s+a}\right]$$

$$tf(t) = L^{-1}\left[\frac{1}{s+b}\right] - L^{-1}\left[\frac{1}{s+a}\right]$$

$$tf(t) = e^{-bt} - e^{-at}$$

$$f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

② $\cot^{-1}(s/a)$

Sol^n: Let $\bar{f}(s) = \cot^{-1}(s/a)$

D.w.r.to s

$$\bar{f}'(s) = \frac{-1}{1+(s/a)^2} \times \frac{d}{ds}(s/a)$$

$$= \frac{-1}{1+s^2/a^2} \times \frac{1}{a}$$

$$= \frac{-1}{\frac{a^2+s^2}{a^2}} \times \frac{1}{a}$$

$$= \frac{-\cancel{a}^2}{a^2+s^2} \times \frac{1}{\cancel{a}}$$

$$\bar{f}'(s) = \frac{-a}{a^2+s^2}$$

$\times$'ly B.S by -1

$$-\bar{f}'(s) = \frac{a}{a^2+s^2} \Rightarrow -\bar{f}'(s) = \frac{a}{s^2+a^2}$$

$$\mathcal{L}^{-1}[-\bar{f}'(s)] = \mathcal{L}^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$t f(t) = \sin at$$

$$f(t) = \frac{\sin at}{t}$$

③ $\log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]$

Sol^n: let $\bar{f}(s) = \log\left[\frac{s^2+4}{s(s+4)(s-4)}\right]$

$\log m - \log n$

$\bar{f}(s) = [\log(s^2+4)] - \{\log\{s(s+4)(s-4)\}]$

$\log mnp = \log m + \log n + \log p$

$= \log(s^2+4) - \{\log s + \log(s+4) + \log(s-4)\}$

$\bar{f}(s) = \log(s^2+4) - \log s - \log(s+4) - \log(s-4)$

D. w.r. to s

$\bar{f}'(s) = \frac{1}{s^2+4}\frac{d}{ds}(s^2+4) - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4}$

$\bar{f}'(s) = \frac{1}{s^2+4} \times 2s - \frac{1}{s} - \frac{1}{s+4} - \frac{1}{s-4}$

$\times^{ly}$ B.S by -1

$-\bar{f}'(s) = -\frac{2s}{s^2+4} + \frac{1}{s} + \frac{1}{s+4} + \frac{1}{s-4}$

$L^{-1}[-\bar{f}'(s)] = -2L^{-1}\left[\frac{s}{s^2+4}\right] + L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s-4}\right]$

$L^{-1}\left[\frac{s}{s^2+2^2}\right]$

$t f(t) = -2\cos 2t + 1 + e^{-4t} - e^{4t}$

$f(t) = \frac{1 + e^{-4t} - e^{4t} - 2\cos 2t}{t}$

(4) $\cot^{-1}\left(\frac{s+a}{b}\right)$

Sol^n, Let $\bar{f}(s) = \cot^{-1}\left(\frac{s+a}{b}\right)$

$\bar{f}'(s) = \frac{-1}{1+\left(\frac{s+a}{b}\right)^2} \times \frac{d}{ds}\left(\frac{s+a}{b}\right)$

$\bar{f}'(s) = \frac{-1}{1+\frac{(s+a)^2}{b^2}} \times \left[\frac{b(1+0)-(s+a)(0)}{b^2}\right]$

$= \frac{-b^2}{b^2+(s+a)^2}\left[\frac{b-0}{\cancel{b^2}}\right]$

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$$\bar{f}'(s) = \frac{-b}{(s+a)^2+b^2}$$

x'ly b.s by -1

$$-\bar{f}'(s) = \frac{b}{(s+a)^2+b^2}$$

$$L^{-1}\left[-\bar{f}'(s)\right] = L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$t\,f(t) = e^{-at}\,L^{-1}\left[\frac{b}{s^2+b^2}\right]$$

$$t\,f(t) = e^{-at}\sin bt$$

$$f(t) = \frac{e^{-at}\sin bt}{t}$$

Convolution theorem

Definition: The convolution of two functions $f(t)$ & $g(t)$ usually denoted by $f(t) * g(t)$ is defined in the form of an integral as follows

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Property ①: $f(t) * g(t) = g(t) * f(t)$ is exist.

i.e to say that convolution operation * is commutative.

Convolution Theorem:

Statement: If $L^{-1}[\bar{f}(s)] = f(t)$ and

$$L^{-1}[\bar{g}(s)] = g(t)$$

then $$L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

Computation of the inverse transform by using convolution theorem:

Working procedure:

① The given function is expressed as product of two functions say $\bar{f}(s)$ & $\bar{g}(s)$

② We find $L^{-1}[\bar{f}(s)] = f(t)$ & $L^{-1}[\bar{g}(s)] = g(t)$

③ We apply convolution theorem in one of the form.

$$L^{-1}[\bar{f}(s)\, \bar{g}(s)] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

④ We evaluate the convolution integral to obtain the required inverse.

Using convolution theorem obtain the inverge Laplace tranfform of the following functions.

① $\frac{1}{s(s^2+a^2)}$

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Soln: Let $\bar{f}(s) = 1/s$ & $\bar{g}(s) = \frac{1}{s^2+a^2}$

Taking inverse

$L^{-1}[\bar{f}(s)] = L^{-1}[1/s] = 1$ and $L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$

$\therefore f(t) = 1$ $g(t) = \frac{1}{a}\sin at$

we have convolution theorem

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_{u=0}^{t} 1\cdot\frac{\sin a(t-u)}{a}\,du$$

$$= \int_{u=0}^{t} \frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{a}\left[\int_{u=0}^{t}\sin(at-au)\,du\right]$$

$$= \frac{1}{a}\left[\frac{-\cos(at-au)}{-a}\right]_{u=0}^{t}$$

$$= \left[\frac{\cos(at-au)}{a^2}\right]_{u=0}^{t}$$

$$= \frac{1}{a}\left(\cos(at-a\cancel{t}) - \cos at\right)$$

$$= \frac{1}{a^2}\left(\cos(0) - \cos at\right)$$

$= \frac{1}{a^2}(1-\cos at)$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)$$

(5) $\frac{s}{(s^2+a^2)^2}$

Dec 2016

Solⁿ: Let $\bar{f}(s) = \frac{1}{s^2+a^2}$ $\bar{g}(s) = \frac{s}{s^2+a^2}$

$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$

$f(t) = \frac{\sin at}{a}$

and $L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right]$

$g(t) = \cos at$

we have convolution theorem

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_{u=0}^{t} f(u)g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s^2+a^2}\times\frac{s}{s^2+a^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a}\cdot\cos(at-au)\,du$$

w.k.T $\sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

where $A = au$, $B = at-au$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{a}\int_{u=0}^{t}\frac{1}{2}\left[\sin(au+at-au)+\sin(au-(at-au))\right]du$$

$$= \frac{1}{2a}\int_{u=0}^{t}(\sin at+\sin(2au-at))\,du$$

w.k.T ~~$\sin(-\theta) = -\sin\theta$~~

~~$= \frac{1}{2}\int_{u=0}^{t}(\sin at - \sin at)\,du$~~

$$= \frac{1}{2a}\int_{u=0}^{t}(\sin at+\sin(2au-at))\,du$$

$$= \frac{1}{2a}\left[\int_{u=0}^{t}\sin at\,du + \int_{u=0}^{t}\sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left\{\left[+\sin at\int_{u=0}^{t}1\,du\right]+\left[-\frac{\cos(2au-at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2a}\left[\sin at\,[u]_{u=0}^{t} - \left\{\frac{\cos(2at-at)}{2a}-\frac{\cos(0-at)}{2a}\right\}\right]$$

$$= \frac{1}{2a}\left[\sin at(t-0) - \frac{\cos(at)}{2a}+\frac{\cos(-at)}{2a}\right]$$

$$= \frac{1}{2a}\left[\sin at\,(t) - \cancel{\frac{\cos at}{2a}} + \cancel{\frac{\cos at}{2a}}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}$$

③ June 2017 $\frac{1}{(s^2+a^2)^2}$

Soln: Let $\bar{f}(s) = \frac{1}{s^2+a^2}$ & $\bar{g}(s) = \frac{1}{s^2+a^2}$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] \;\&\; L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$f(t) = \frac{\sin at}{a}, \quad g(t) = \frac{\sin at}{a}$$

we have convolution theorem

$$L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)} \times \frac{1}{(s^2+a^2)}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \times \frac{\sin a(t-u)}{a}\,du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} \frac{\sin au}{a} \times \frac{\sin(at-au)}{a}\,du$$

$$= \frac{1}{a^2}\int_{u=0}^{t} \sin au \,\sin(at-au)\,du$$

W.K.T $\quad -2\sin A\sin B = \cos(A+B) - \cos(A-B)$

$$\sin A\sin B = -\frac{1}{2}\left[\cos(A+B) - \cos(A-B)\right]$$

$$= \frac{1}{a^2}\int_{u=0}^{t} -\frac{1}{2}\left[\cos(au+at-au) - \cos(au-(at-au))\right]du$$

$$= -\frac{1}{2a^2}\int_{u=0}^{t} \left(\cos at - \cos(2au - at)\right) du$$

$$= -\frac{1}{2a^2}\left[\int_{u=0}^{t} \cos at \cdot du - \int_{u=0}^{t} \cos(2au - at)\, du\right]$$

$$= -\frac{1}{2a^2}\left[\cos at \int_{u=0}^{t} 1 \cdot du - \left[\frac{\sin(2au-at)}{2a}\right]_{u=0}^{t}\right]$$

$$= -\frac{1}{2a^2}\left[\cos at \left[u\right]_{u=0}^{t} - \left\{\frac{\sin(2at-at)}{2a} - \frac{\sin(0-at)}{2a}\right\}\right]$$

$$= -\frac{1}{2a^2}\left[\cos at\,[t-0] - \left\{\frac{\sin at}{2a} - \frac{\sin(-at)}{2a}\right\}\right]$$

$$= -\frac{1}{2a^2}\left[t\cos at - \left\{\frac{\sin at}{2a} - \times - \frac{\sin at}{2a}\right\}\right]$$

$$= -\frac{1}{2a^2}\left[t\cos at - \frac{\sin at}{2a} - \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^2}\left[-t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2a^2}\left[-t\cos at + \frac{\cancel{2}\sin at}{\cancel{2}a}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^2}\left[-t\cos at + \sin at\right]$$

(H)

Dec 2017

Sol'

(H) $\dfrac{1}{(s-1)(s^2+1)}$

Dec 2017

<u>Soln:</u> Let $\bar{f}(s) = \frac{1}{s-1}$ & $\bar{g}(s) = \frac{1}{s^2+1}$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s-1}\right] = e^t$$

$$\therefore f(t) = e^t$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

$$\therefore g(t) = \sin t$$

Now we convolution theorem

$$L^{-1}[\bar{f}(s)\,\bar{g}(s)] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \int_{u=0}^{t} e^{u} \sin(t-u)\,du$$

w.k.T $\int e^{ax}\sin(bx+c)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c) - b\cos(bx+c)\right]$

$a=1 \quad b=-1$

$$L^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \left[\frac{e^u}{1+1}\left((1)\sin(t-u) - (-1)\cos(t-u)\right)\right]_{u=0}^{t}$$

$$= \left[\frac{e^u}{2}\left(\sin(t-u) + \cos(t-u)\right)\right]_{u=0}^{t}$$

$$= \left[\left\{\frac{e^t}{2}\left(\sin(t-t) + \cos(t-t)\right\} - \left\{\frac{e^0}{2}\left(\sin(t-0) + \cos(t-0)\right)\right\}\right]$$

$$= \left\{\frac{e^t}{2}\left(\sin(0) + \cos(0)\right) - \frac{1}{2}(\sin t + \cos t)\right\}$$

$$= \frac{e^t}{2}(0+1) - \frac{1}{2}(\sin t + \cos t)$$

$= \frac{1}{2}e^{t} - \frac{1}{2}\sin t - \frac{1}{2}\cos t$

$$\mathcal{L}^{-1}\left[\frac{1}{(s-1)(s^2+1)}\right] = \frac{1}{2}\left(e^{t} - \sin t - \cos t\right)$$

Doyourself

⑤ $\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$ using convolution theorem

June 2018

⑥ $\frac{s^2}{(s^2+a^2)^2}$

Soln: Let $\bar{f}(s) = \frac{s}{s^2+a^2}$ and $\bar{g}(s) = \frac{s}{s^2+a^2}$

$\mathcal{L}^{-1}[\bar{f}(s)] = \cos at$ and $\mathcal{L}^{-1}[\bar{g}(s)] = \cos at$

$f(t) = \cos at$ and $g(t) = \cos at$

Now by applying convolution theorem we have

$$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_{u=0}^{t} f(u)\,g(t-u)\,du$$

$$= \int_{u=0}^{t} \cos au \cos a(t-u)\,du$$

$$= \int_{u=0}^{t} \cos au \cos(at-au)\,du$$

NOTE: $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$= \frac{1}{2}\int_{u=0}^{t} \left\{\cos(au+at-au) + \cos(au-(at-au))\right\}du$$

$\mathcal{L}^{-1}\left[\frac{s^2}{(s^2+ \ldots\right.$

$$= \frac{1}{2}\int_{u=0}^{t} (\cos at + \cos(at - at + au))\, du$$

$$= \frac{1}{2}\int_{u=0}^{t} (\cos at + \cos(2au - at))\, du$$

$$= \frac{1}{2}\left[\int_{u=0}^{t} \cos at\, du + \int_{u=0}^{t} \cos(2au - at)\, du\right.$$

$$= \frac{1}{2}\left\{\cos at \int_{u=0}^{t} 1\, du + \int_{u=0}^{t} \cos(2au - at)\, du\right\}$$

$$= \frac{1}{2}\left\{\cos at\, [u]_{u=0}^{t} + \left[+\frac{\sin(2au - at)}{2a}\right]_{u=0}^{t}\right\}$$

$$= \frac{1}{2}\left\{\cos at\,(t - 0) + \frac{1}{2a}\left\{\sin(2at - at) - \sin(2a(0) - at)\right\}\right\}$$

$$= \frac{1}{2}\left\{\cos at \cdot t + \frac{1}{2a}\left(\sin(at) - \sin(0 - at)\right)\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{2a}(\sin at - \sin(-at))\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{2a}(\sin at + \sin at)\right\}$$ W.K.T $\sin(-\theta) = -\sin\theta$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{\cancel{2}a}(\cancel{2}\sin at)\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{a}\sin at\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \sin at\right\}$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}\left\{at\cos at + \sin at\right\}$$

$\boxed{at - au))\}du}$

Do yourself

(7) $\frac{1}{s^2(s+1)^2}$

Solⁿ: Let $\bar{f}(s) = 1/s^2$ $\quad \bar{g}(s) = 1/(s+1)^2$

$L^{-1}[\bar{f}(s)] = L^{-1}[1/s^2]$ and $L^{-1}[\bar{g}(s)] = 1/(s+1)^2$

$f(t) = t$ and $g(t) = e^{-t} \cdot t$

Now by applying Convolution theorem we have

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = \int_{u=0}^{t} f(u)\, g(t-u)\, du$$

$$= \int_{u=0}^{t} u\, e^{-(t-u)} (t-u)\, du$$

$$= \int_{u=0}^{t} u \cdot e^{-t+u} \cdot (t-u)\, du$$

$$= \int_{u=0}^{t} u \cdot e^{-t} \cdot e^{u} (t-u)\, du$$

$$= e^{-t} \int_{u=0}^{t} (tu - u^2)\, e^{u}\, du$$

Apply Bernoulli's rule

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = e^{-t}\left[(tu-u^2)e^{u} - (t-2u)e^{u} + (0-2)e^{u} \right]_{u=0}^{t}$$

$$= e^{-t}\left[\{ \cancel{(t \cdot t - t^2)}^{0} e^{t} - (t-2t)e^{t} + (-2)e^{t} \} - \{ (0-0)e^{0} - (t-0)e^{0} + (-2)e^{0} \} \right]$$

$= \bar{e}^{t}\left[0 + (-t)e^{t} - 2e^{t} - 0 + t + 2\right]$

$= \bar{e}^{t}\left[\cancel{t}e^{t} - 2e^{t} + t + 2\right]$

$= \bar{e}^{t}\,t\,e^{t} - 2e^{t}\cdot\bar{e}^{t} + t\bar{e}^{t} + 2\bar{e}^{t}$

$= t - 2 + \bar{e}^{t}(2+t)$

(or)

$$L^{-1}\left[\frac{1}{s^2(s+1)}\right] = 2(e^{-t} - 1) + t(1 + \bar{e}^{t})$$

Solution of linear differential equations using Laplace transforms (Initial value problems)

Laplace transform of the derivatives

We derive an expression for $L[y'(t)]$ and hence we deduce the expression for $L[y''(t)]$, $L[y'''(t)]$

so, $L[y'(t)] = sL[y(t)] - y(0)$

$L[y''(t)] = s^2 L[y(t)] - sy(0) - y'(0)$

$L[y'''(t)] = s^3 L[y(t)] - s^2 y(0) - sy'(0) - y''(0)$

Working procedure

1. The given differential eqn is expressed in the notation $y'(t), y''(t), y'''(t) \ldots$ for the derivatives
2. We take Laplace transform on both sides of given equation
3. We use the expressions for $L[y'(t)], L[y''(t)] \ldots$

④ we substitute the given initial conditions and simplify to obtain $\mathcal{L}[y(t)]$ as a function of s.

⑤ we find the inverse to obtain $y(t)$

Problems

① Solve by using Laplace transforms

$$\frac{d^2y}{dt^2} + k^2 y = 0$$ given that $y(0) = 2$, $y'(0) = 0$

Dec 2018

Soln: The given eqn is $y''(t) + k^2 y(t) = 0$

Taking Laplace transform on B.S

$$\mathcal{L}[y''(t) + k^2 y(t)] = \mathcal{L}[0]$$

~~Taking Laplace~~

$$\{s^2 \mathcal{L}[y(t)] - s y(0) - y'(0)\} + k^2 \mathcal{L}[y(t)] = 0$$

using the given initial conditions we obtain

$$(s^2 + k^2)\mathcal{L}[y(t)] - s y(0) - y'(0) = 0$$

$$(s^2 + k^2)\mathcal{L}[y(t)] - s(2) - 0 = 0$$

$$(s^2 + k^2)\mathcal{L}[y(t)] - 2s = 0$$

$$\mathcal{L}[y(t)] = \frac{2s}{s^2 + k^2}$$

$$[y(t)] = 2\mathcal{L}^{-1}\left[\frac{s}{s^2 + k^2}\right]$$

w.k.t $\mathcal{L}^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos at$

$$\therefore y(t) = 2\cos kt$$

(2) Solve $y''' + 2y'' - y' - 2y = 0$ given $y(0) = y'(0) = 0$ and $y''(0) = 6$ by using Laplace transform method.

Soln: $y''' + 2y'' - y' - 2y = 0$ — (1)

Given initial condition $y(0) = y'(0) = 0$ & $y''(0) = 6$

Taking Laplace transform on B.S of (1)

$L[y'''(t)] + 2L[y''(t)] - L[y'(t)] - 2L[y(t)] = L[0]$

$\{s^3 L[y(t)] - s^2 y(0) - s y'(0) - y''(0)\} +$
$2\{s^2 L[y(t)] - s y(0) - y'(0)\} - \{s L[y(t)] - y(0)\}$
$-2L[y(t)] = 0$

$L[y(t)]\{s^3 + 2s^2 - s - 2\} - s^2 y(0) - s y'(0) - y''(0)$
$-2s y(0) - 2y'(0) + y(0) = 0$

$L[y(t)]\{s^2(s+2) - 1(s+2)\} - s^2(0) - s(0) - 6 - 2(0)$
$-2(0) + 0 = 0$

$L[y(t)]\{(s+2)(s^2-1)\} - 0 - 0 - 6 - 0 - 0 = 0$

$L[y(t)]\{(s+2)(s^2-1)\} - 6 = 0$

$a^2 - b^2 = (a+b)(a-b)$

$L[y(t)]\{(s+2)(s+1)(s-1)\} = 6$

$$L[y(t)] = \frac{6}{(s+2)(s-1)(s+1)}$$

$$y(t) = L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] \quad — (*)$$

By partial fractions we have to solve

$$\frac{6}{(s+2)(s-1)(s+1)} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{s+1} \quad — (1)$$

x^{ly} B.S by $(s+2)(s-1)(s+1)$

$6 = A(s-1)(s+1) + B(s+2)(s+1) + C(s+2)(s-1)$

put $s=-2$

$6 = A(-3)(-1) + B(0) + C(0)$

$6 = 3A \Rightarrow \underline{\underline{A=2}}$

put $s=1$

$6 = A(0) + B(3)(2) + C(0)$

$6 = 0 + 6B + 0 \Rightarrow 6B = 6 \Rightarrow B = 1$

put $s=-1$

$6 = A(-2)(0) + B(0) + C(1)(-2)$

$6 = 0 + 0 - 2C$

$-2C = 6 \Rightarrow \underline{C=-3}$

put A, B, C in (1)

$$\frac{6}{(s+2)(s-1)(s+1)} = \frac{2}{s+2} + \frac{1}{s-1} + \frac{-3}{s+1}$$

$$L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] = 2L^{-1}\left[\frac{1}{s+2}\right] + L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{s+1}\right]$$

$$L^{-1}\left[\frac{6}{(s+2)(s-1)(s+1)}\right] = 2e^{-2t} + e^{t} - 3e^{-t}$$

(*) becomes

$$\underline{\underline{y(t) = 2e^{-2t} + e^{t} - 3e^{-t}}}$$

$\cancel{L^{-1}\{f(s)\} = f(t)}$

③ Solve the following initial value problem by using Laplace transform

June 2016 $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$; $y(0) = 0, y'(0) = 0$

Soln: Given $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$

$y(0) = 0, \; y'(0) = 0$

$y''(t) + 4y'(t) + 4y(t) = e^{-t}$

Taking Laplace transform on both sides

$L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[e^{-t}]$

$\{s^2 L[y(t)] - sy(0) - y'(0)\} + 4\{sL[y(t)] - y(0)\}$

$+ 4L[y(t)] = \frac{1}{s+1}$

Using the given initial conditions we obtain

$L[y(t)]\{s^2 + 4s + 4\} - sy(0) - y'(0) - 4y(0)$

$= \frac{1}{s+1}$

$L[y(t)]\{s^2 + 4s + 4\} - s(0) - 0 - 4(0) = 1/s+1$

$L[y(t)]\{s^2 + 4s + 4\} - 0 - 0 - 0 = 1/s+1$

$L[y(t)]\{(s+2)^2\} = 1/s+1$

$L[y(t)] = \frac{1}{(s+1)(s+2)^2}$

$y(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right]$ — ⊛

$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$ — ①

$\times^{ly}$ B.S by $(s+1)(s+2)^2$

$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1)$ — ②

put $s=-1$

$1 = A(-1+2) + B(0) + C(0)$

$1 = A(1) \Rightarrow A = 1$

put $s=-2$

$1 = A(0) + B(0) + C(-1)$

$1 = 0 + 0 - C$

$C = -1$

equating the co-efficient of s^2 on B.S of ②

$0 = A + B$

W.K.T $A = 1$

$B = -1$

$1 = A(s^2+4+4s) + B(s^2+3s+2) + C(s+1)$

In LHS there is no s^2

so co-eff of s^2 is zero

$0 = A + B$

$A = 1$

$B = -1$

hence

① $\Rightarrow$

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= e^{-t} - e^{-2t} - e^{-2t}\cdot\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \quad (\because a=-2)$$

$$= e^{-t} - e^{-2t} - e^{-2t}\cdot t$$

$$\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s+2)^2}\right] = e^{-t} - e^{-2t}(1+t)$$

(*) becomes

$$y(t) = e^{-t} - e^{-2t}(1+t)$$

(ii) Employ laplace transform to solve the equation $y''+5y'+6y=5e^{2x}$, $y(0)=2, y'(0)=1$

Sol^n: Given $y''(x)+5y'(x)+6y(x)=5e^{2x}$ —— (1)

$y(0)=2, \; y'(0)=1$

Taking laplace transform on both sides of (1)

$$\mathcal{L}[y''(x)]+5\mathcal{L}[y'(x)]+6\mathcal{L}[y(x)]=5\mathcal{L}[e^{2x}]$$

$$\{s^2\mathcal{L}[y(x)]-sy(0)-y'(0)\}+5\{s\mathcal{L}[y(x)]-y(0)\}+6\mathcal{L}[y(x)]=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{s^2+5s+6\}-sy(0)-y'(0)-5y(0)=\frac{5}{s-2}$$

Use the initial condition

$$\mathcal{L}[y(x)]\{s^2+5s+6\}-s(2)-1-5(2)=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{s^2+5s+6\}-2s-11=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{(s+2)(s+3)\}-(2s+11)=\frac{5}{s-2}$$

$$\mathcal{L}[y(x)]\{(s+2)(s+3)\}=\frac{5}{s-2}+(2s+11)$$

$$\mathcal{L}[y(x)]\{(s+2)(s+3)\}=\frac{5+(2s+11)(s-2)}{s-2}$$

$$\mathcal{L}[y(x)]=\frac{5+2s^2+11s-4s-22}{(s+2)(s+3)(s-2)}$$

$$\mathcal{L}[y(x)]=\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}$$

$$y(x)=\mathcal{L}^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] \quad \text{—— } (*)$$

④ Employ laplace transform to solve the equation $y'' + 5y' + 6y = 5e^{2x}$, $y(0) = 2$, $y'(0) = 1$

Sol^n: Given $y''(x) + 5y'(x) + 6y(x) = 5e^{2x}$ — ①

$y(0) = 2, \; y'(0) = 1$

Taking laplace transform on both sides of ①

$$L[y''(x)] + 5L[y'(x)] + 6L[y(x)] = 5L[e^{2x}]$$

$$\{s^2 L[y(x)] - sy(0) - y'(0)\} + 5\{sL[y(x)] - y(0)\} + 6L[y(x)] = \frac{5}{s-2}$$

$$L[y(x)]\{s^2 + 5s + 6\} - sy(0) - y'(0) - 5y(0) = \frac{5}{s-2}$$

Use the initial condition

$$L[y(x)]\{s^2 + 5s + 6\} - s(2) - 1 - 5(2) = \frac{5}{s-2}$$

$$L[y(x)]\{\underbrace{s^2 + 5s + 6}_{\downarrow}\} - 2s - 11 = \frac{5}{s-2}$$

$$L[y(x)]\{(s+2)(s+3)\} - (2s+11) = \frac{5}{s-2}$$

$$L[y(x)]\{(s+2)(s+3)\} = \frac{5}{s-2} + (2s+11)$$

$$L[y(x)]\{(s+2)(s+3)\} = \frac{5 + (2s+11)(s-2)}{s-2}$$

$$L[y(x)] = \frac{5 + 2s^2 + 11s - 4s - 22}{(s+2)(s+3)(s-2)}$$

$$L[y(x)] = \frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)}$$

$$y(x) = L^{-1}\left[\frac{2s^2 + 7s - 17}{(s-2)(s+2)(s+3)}\right] \quad \text{— (*)}$$

$$\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s+3} \quad \text{--- (1)}$$

$\times^{ly}$ B.S by $(s-2)(s+2)(s+3)$

$$2s^2+7s-17 = A(s+2)(s+3) + B(s-2)(s+3) + C(s-2)(s+2)$$

put $s=2$

$$2(2)^2+7(2)-17 = A(4)(5) + B(0) + C(0)$$

$$8+14-17 = 20A + 0 + 0$$

$$5 = 20A \Rightarrow \underline{\underline{A = 1/4}}$$

put $s=-2$

$$2(-2)^2+7(-2)-17 = A(0) + B(-2-2)(-2+3) + C(0)$$

$$2(4)-14-17 = 0 + B(-4)(1) + 0$$

$$+23 = +4B$$

$$\underline{\underline{B = 23/4}}$$

put $s=-3$

$$2(-3)^2+7(-3)-17 = A(0) + B(0) + C(-3-2)(-3+2)$$

$$2(9)-21-17 = 0+0+C(-5)(-1)$$

$$18-21-17 = 5C$$

$$-20 = 5C \Rightarrow C = \frac{-20}{5}$$

$$\underline{\underline{C = -4}}$$

(1) $\Rightarrow$

$$\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)} = \frac{1}{4(s-2)} + \frac{23}{4(s+2)} + \frac{-4}{(s+3)}$$

$$\mathcal{L}^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + \frac{23}{4}\mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$\mathcal{L}^{-1}\left[\frac{2s^2+7s-17}{(s-2)(s+2)(s+3)}\right] = \frac{1}{4}e^{2x} + \frac{23}{4}e^{-2x} - 4e^{-3x}$$

(*) becomes

$$y(x) = \frac{1}{4}e^{2x} + \frac{23}{4}e^{-2x} - 4e^{-3x}$$

(5) June 2017 Using laplace tranyform technique solve $x'' - 2x' + x = e^{2t}$ with $x(0) = 0,\ x'(0) = -1$

Soln: Given $x'' - 2x' + x = e^{2t}$

$$x''(t) - 2x'(t) + x(t) = e^{2t}$$

Take laplace tranyform on B.S

$$\mathcal{L}[x''(t)] - 2\mathcal{L}[x'(t)] + \mathcal{L}[x(t)] = \mathcal{L}[e^{2t}]$$

$$\{s^2\mathcal{L}[x(t)] - sx(0) - x'(0)\} - 2\{s\mathcal{L}[x(t)] - x(0)\} + \mathcal{L}[x(t)] = 1/s-2$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} - sx(0) - x'(0) - 2x(0) = 1/s-2$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} - s(0) - (-1) - 2(0) = 1/s-2$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} - 0 + 1 - 0 = 1/s-2$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} + 1 = \frac{1}{s-2}$$

$$\mathcal{L}[x(t)]\{s^2 - 2s + 1\} = \frac{1}{s-2} - 1$$

$[x(t)]\{s^2-2s+1\} = \frac{1-s+2}{s-2}$

$$\mathcal{L}[x(t)] = \frac{3-s}{(s-2)(s^2-2s+1)}$$

$$\mathcal{L}[x(t)] = \frac{3-s}{(s-2)(s-1)^2}$$

$$\mathcal{L}^{-1} \quad x(t) = \mathcal{L}^{-1}\left[\frac{3-s}{(s-2)(s-1)^2}\right] \quad — (*)$$

$$\frac{3-s}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \quad — (1)$$

$\times^{ly}$ $(s-2)(s-1)^2$

$$3-s = A(s-1)^2 + B(s-1)(s-2) + C(s-2) \quad — (2)$$

put $s=2$

$$3-2 = A(2-1)^2 + B(0) + C(0)$$

$$1 = A(1) + 0 + 0 \Rightarrow A = 1$$

put $s=1$

$$3-1 = A(0) + B(0) + C(-1)$$

$$2 = -C \Rightarrow \underline{C=-2}$$

equating the Co-efficient of s^2 on both sides of (2)

There is no s^2 in the LHS so

$$0 = A(s^2-2s+1) + B(s^2-3s+2) + C(s-2)$$

$$0 = A+B$$

$$0 = A + B$$

w.k.T $A=1$

$$0 = 1 + B \Rightarrow \underline{B=-1}$$

put A, B, C values in (1)

$$\frac{3-s}{(s-2)(s-1)^2} = \frac{1}{s-2} - \frac{1}{s-1} - \frac{2}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left[\frac{3-s}{(s-2)(s-1)^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= e^{2t} - e^{t} - 2e^{t} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{2t} - e^{t} - 2e^{t}\, t$$

$$\mathcal{L}^{-1}\left[\frac{3-s}{(s-2)(s-1)^2}\right] = e^{2t} - e^{t}(1+2t)$$

$$(*) \Rightarrow x(t) = e^{2t} - e^{t}(1+2t)$$

Doyourself

June 2015 (6) Solve by using Laplace transform of

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = te^{-t} \text{ with } y(0) = 1,\ y'(0) = -2$$

(7) Solve $y'' + 6y' + 9y = 12t^2e^{-3t}$ subject to the conditions $y(0) = 0 = y'(0)$ by using the Laplace transforms.

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Soln: the given equation is

$$y''(t) + 6y'(t) + 9y(t) = 12t^2e^{-3t}$$

Take Laplace transform on B.S

$$\mathcal{L}[y''(t)] + 6\mathcal{L}[y'(t)] + 9\mathcal{L}\{y(t)\} = 12\mathcal{L}[t^2e^{-3t}]$$

$$\{s^2\mathcal{L}[y(t)] - sy(0) - y'(0)\} + 6\{s\mathcal{L}[y(t)] - y(0)\}$$

$$+ 9\mathcal{L}[y(t)] = 12\cdot\mathcal{L}[t^2]_{s \to s+3}$$

$$\mathcal{L}[y(t)]\{s^2 + 6s + 9\} - sy(0) - y'(0) - 6y(0) = 12\left[\frac{2!}{s^3}\right]_{s \to s+3}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[y(t)]\{s^2+6s+9\} - 0 - 0 - 0 = \frac{24}{(s+3)^3}$$

$$\mathcal{L}[y(t)]\{(s+3)^2\} = \frac{24}{(s+3)^3}$$

$$\mathcal{L}[y(t)] = \frac{24}{(s+3)^3 (s+3)^2}$$

$$\mathcal{L}[y(t)] = \frac{24}{(s+3)^5}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{24}{(s+3)^5}\right]$$

$$= 24\, \mathcal{L}^{-1}\left[\frac{1}{(s+3)^5}\right]$$

$$= 24\, e^{-3t}\, \mathcal{L}^{-1}\left[\frac{1}{s^5}\right]$$

$$= 24 e^{-3t} \cdot \frac{t^4}{4!}$$

$$= \cancel{24} \cdot e^{-3t} \cdot \frac{t^4}{\cancel{24}}$$

$$\boxed{y(t) = e^{-3t} \cdot t^4}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^5}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^{4+1}}\right] = \frac{t^4}{4!} \quad (\because n=4)$$

(8) Solve the following boundary value problem using Laplace transforms
$y''(t) + y(t) = 0$; $y(0) = 2$, $y(\pi/2) = 1$

Soln: $y''(t) + y(t) = 0$
Take Laplace transform on both sides
$$\mathcal{L}[y''(t)] + \mathcal{L}[y(t)] = \mathcal{L}[0]$$
$$\mathcal{L}[y''(t)] + \mathcal{L}[y(t)] = 0 \quad \text{—} (1)$$
$$\{s^2 \mathcal{L}[y(t)] - s y(0) - y'(0)\} + \mathcal{L}[y(t)]$$
Let us assume $y'(0) = a$, where a is a constant to be found later & we have $y(0) = 2$ by data
hence eqn (1) becomes

$$(s^2+1)\mathcal{L}[y(t)]-2s-a=0$$

$$(s^2+1)\mathcal{L}[y(t)]=2s+a$$

$$\mathcal{L}[y(t)]=\frac{2s+a}{s^2+1}$$

$$\mathcal{L}^{-1}\quad \mathcal{L}[y(t)]=\frac{2s+a}{s^2+1}$$

$$y(t)=\mathcal{L}^{-1}\left[\frac{2s+a}{s^2+1}\right]$$

$$=\mathcal{L}^{-1}\left[\frac{2s}{s^2+1}\right]+\mathcal{L}^{-1}\left[\frac{a}{s^2+1}\right]$$

$$=2\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+a\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t)=2\cos t+a\sin t \quad - (*)$$

now we shall use the condition $y(\pi/2)=1$

$$y(\pi/2)=2\cos\pi/2+a\sin\pi/2$$

$$1=2(0)+a(1)$$

$$1=0+a$$

$$\underline{\underline{a=1}}$$

$$(*) \Rightarrow \underline{\underline{y(t)}}=2\cos t+\sin t$$

yourself

Solve the DE $y''+4y'+3y=e^{-t}$ with $y(0)=1=y'(0)$ using laplace transforms.

ans: $y(t)=\frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}.t-\frac{3}{4}e^{-3t}$

by partial fractions we have to solve

Soln: Given $y''+4y'+3y=e^{-t}$

$y''(t)+4y'(t)+3y(t)=e^{-t}$

Taking laplace transform on both sides

$L[y''(t)]+4L[y'(t)]+3L[y(t)]=L[e^{-t}]$

$\{s^2L[y(t)]-sy(0)-y'(0)\}+4\{sL[y(t)]-y(0)$

$+3L[y(t)]=\frac{1}{s+1}$

$L[y(t)]\{s^2+4s+3\}-sy(0)-y'(0)-4y(0)$

$=\frac{1}{s+1}$

Use initial conditions

$L[y(t)]\{s^2+4s+3\}-s(1)-(1)-4(1)=1/s+1$

$L[y(t)]\{s^2+3s+s+3\}-s-5=1/s+1$

$L[y(t)]\{(s+3)(s+1)\}-(s+5)=1/s+1$

$L[y(t)]\{(s+3)(s+1)\}=\frac{1}{s+1}+s+5$

$L[y(t)]\{(s+3)(s+1)\}=\frac{1+(s+5)(s+1)}{(s+1)}$

$L[y(t)]\{(s+3)(s+1)\}=\frac{1+s^2+s+5s+5}{(s+1)}$

$L[y(t)]\{(s+3)(s+1)\}=\frac{s^2+6s+6}{(s+1)}$

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$$\mathcal{L}[y(t)] = \frac{s^2+6s+6}{(s+1)(s+1)(s+3)}$$

$$\mathcal{L}[y(t)] = \frac{s^2+6s+6}{(s+1)^2(s+3)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] \quad \text{—} \ (*)$$

by partial fractions we have to solve

$$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+3} \quad \text{—} \ (1)$$

$\times^{ly}$ B.S by $(s+1)^2(s+3)$

$$s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 \quad \text{—}$$

put $s=-1$

$$(-1)^2 \cancel{-6} + \cancel{6} = A(0) + B(-1+3) + C(0)$$

$$1 = 0 + 2B + 0$$

$$2B = 1 \Rightarrow B = 1/2$$

put $s=-3$

$$(-3)^2 + 6(-3) + 6 = A(-3+1)(0) + B(0) + C(-3+1)^2$$

$$9 - 18 + 6 = 0 + 0 + C(-2)^2$$

$$-3 = 4C$$

$$C = -3/4$$

equating the co-efficient of s^2 on B.S of (2)

$$1 = A + C$$

$$1 = A - 3/4 \Rightarrow A = 1 + 3/4$$

$$A = 7/4$$

A, B, C values in (1)

$$\frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{7}{4(s+1)} + \frac{1}{2(s+1)^2} - \frac{3}{4(s+3)}$$

$$L^{-1}\left[\frac{s^2+6s+6}{(s+1)^2(s+3)}\right] = \frac{7}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4}L^{-1}\left[\frac{1}{s+3}\right]$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}L^{-1}\left[\frac{1}{s^2}\right] - \frac{3}{4}e^{-3t}$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\,t - \frac{3}{4}e^{-3t}$$

$$= \frac{7}{4}e^{-t} + \frac{1}{2}$$

$\therefore$ (*) becomes

$$y(t) = \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}\,t - \frac{3}{4}e^{-3t}$$

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(6) $y''(t) + 2y'(t) + y(t) = te^{-t}$

$$L[y''(t)] + 2L[y'(t)] + L[y(t)] = L[te^{-t}]$$

$$\{s^2L[y(t)] - sy(0) - y'(0)\} + 2\{sL[y(t)] - y(0)\}$$
$$+ L[y(t)] = L[t]_{s\to s+1}$$

$$L[y(t)]\{s^2+2s+1\} - sy(0) - y'(0) - 2y(0)$$
$$= \left[\frac{1!}{s^2}\right]_{s\to s+1}$$

$$L[y(t)]\{(s+1)^2\} - s(1) - \cancel{(-2)} - \cancel{2(1)} = \frac{1}{(s+1)^2}$$

$$L[y(t)]\{(s+1)^2\} - s = \frac{1}{(s+1)^2}$$

$$L[y(t)]((s+1)^2) = \frac{1}{(s+1)^2} + s$$

$$L\,\cancel{[y(t)](s+1)^2} = \cancel{\frac{1+s(s+1)^2}{(s+1)^2}}$$

$$L\,\cancel{[y(t)]} = \cancel{\frac{1+s(s^2+1+2s)}{(s+1)^2(s+1)^2}} = \frac{1+s^3+s+2s^2}{(s+1)^4}$$

$$L[y(t)] = \frac{1}{(s+1)^4} + \frac{s}{(s+1)^2}$$

$$y(t) = L^{-1}\left[\frac{1}{(s+1)^4}\right] + L^{-1}\left[\frac{s}{(s+1)^2}\right]$$

$$= e^{-t} L^{-1}\left[\frac{1}{s^4}\right] + L^{-1}\left[\frac{(s+1)-1}{(s+1)^2}\right]$$

$$= e^{-t} \cdot \frac{t^3}{3!} + L^{-1}\left[\frac{s-1}{s^2}\right]$$

$$= e^{-t} \cdot \frac{t^3}{6} + L^{-1}\left[\frac{\cancel{s}}{s^{\cancel{2}}}\right] - L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-t} \cdot \frac{t^3}{6} + L^{-1}\left[\frac{1}{s}\right] - t$$

$$= e^{-t} \cdot \frac{t^3}{6} + (1) - t$$

$$y(t) = e^{-t} \cdot \frac{t^3}{6} + 1 - t$$